

# Exact Solution of a Cellular Automaton for Traffic

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We present an exact solution of a probabilistic cellular automaton for traffic with open boundary conditions, e.g., cars can enter and leave a part of a highway with certain probabilities. The model studied is the asymmetric exclusion process (ASEP) with *simultaneous* updating of all sites. It is equivalent to a special case ( $v_{\max} = 1$ ) of the Nagel–Schreckenberg model for highway traffic, which has found many applications in real-time traffic simulations. The simultaneous updating induces additional strong short-range correlations compared to other updating schemes. The stationary state is written in terms of a matrix product solution. The corresponding algebra, which expresses a system-size recursion relation for the weights of the configurations, is quartic, in contrast to previous cases, in which the algebra is quadratic. We derive the phase diagram and compute various properties such as density profiles, two-point functions, and the fluctuations in the number of particles (cars) in the system. The current and the density profiles can be mapped onto the ASEP with other time-discrete updating procedures. Through use of this mapping, our results also give new results for these models.

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**KEY WORDS:** Asymmetric exclusion process; boundary-induced phase transitions; steady state; matrix product Ansatz; cellular automaton.

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## 1. INTRODUCTION

In this paper we study a simple probabilistic cellular automaton which describes the flow—of particles, automobiles, or some other conserved quantity—through a one dimensional system. The particles (or cars) of the model move on a finite lattice; at integer times they *simultaneously* attempt to hop one site forward, succeeding with probability  $p$  if the site in front of them is empty. We are interested in the case of open boundary conditions (OBC), in which, simultaneously with the hopping of particles along

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the lattice, a particle enters the system with probability  $\alpha$  at the leftmost site if that site is empty, and if the rightmost site is occupied then the particle on that site exits with probability  $\beta$ . We use a matrix product ansatz to give a complete solution of this model.

In recent years, cellular automata models for traffic flow have gained much attention, because they make real time traffic simulations possible (see ref. 1 and references therein). The model studied here is one such; for example, if the injection probability  $\alpha$  is large the model may be regarded as describing the situation, familiar from everyday experience, of the reduction of a two-lane to a one-lane road by, e.g., the presence of construction work on one lane. It is a special case of the well-known Nagel–Schreckenberg<sup>(2)</sup> model, obtained by requiring that the parameter  $v_{\max}$  of that model satisfies  $v_{\max} = 1$  so that cars move at most one lattice spacing at each integer time. In realistic computer simulations of highway traffic, the Nagel–Schreckenberg model is usually used with  $v_{\max} = 5$ . However, in the case of OBC the phase diagram and density profiles are essentially independent of  $v_{\max}$ ,<sup>(3)</sup> and in general it has been observed that, for modeling city traffic, it is sufficient to set  $v_{\max} = 1$ .<sup>(4)</sup>

The model is in fact a synchronous update version of the asymmetric exclusion process (ASEP), widely studied in both the physics and mathematics literature.<sup>(5–7)</sup> The ASEP was originally introduced as an interacting particle system evolving in continuous time; this evolution is equivalent to the random sequential update (RSU) procedure, in which randomly chosen particles hop one at a time. Other updating schemes have also been introduced, including sublattice-parallel<sup>(8,9)</sup> and ordered sequential procedures,<sup>(10)</sup> and the fully parallel updating (PU) scheme, which corresponds to the probabilistic cellular automaton described above. See Section 9 and ref. 11 for precise definitions and a review of current knowledge about these models. The ASEP with RSU and OBC has been exactly solved,<sup>(12–14)</sup> using (among other methods) the matrix product ansatz, and these results have been extended to the sublattice-parallel and ordered sequential updating schemes.<sup>(15, 10, 16, 11)</sup> The model with PU has proved to be less tractable; for example, parallel updating can induce strong short range correlations<sup>(17)</sup> (these are absent under other updating schemes, see Rajewsky in ref. 1). On the other hand, parallel updating is important in practice for traffic modeling, both for efficiency—the PU is usually much faster than the RSU—and for effectiveness; for example, the Nagel–Schreckenberg model is always implemented with PU, since that method has been found to give the best agreement with measurements on freeway traffic.<sup>(2)</sup>

We remark that if one writes  $\beta = p\tilde{\beta}$  and  $\alpha = p\tilde{\alpha}$  and then takes the limit  $p \rightarrow 0$ , the model reduces to the ASEP with random sequential updating and injection and extraction rates  $\tilde{\beta}$  and  $\tilde{\alpha}$ , respectively.

Let us now briefly discuss known results which are related to our model. For the parallel dynamics on a ring (that is, with periodic boundary conditions), the exact solution was given in ref. 2; here, in contrast to other updating schemes, the stationary state is not a simple product measure—occupation numbers at distinct sites are correlated. For example, if  $p$  is 1 and the density is  $1/2$  then the stationary state consists of free flowing particle-hole pairs, i.e., there is a strong particle-hole attraction. For  $p = 1$ , the dynamics is equivalent to rule 184 for cellular automata, for which transient properties have been analyzed.<sup>(18)</sup> For  $p$  close to 1, and  $\alpha = \beta = 1$  the scaling behavior of the lifetime of stop-start waves has been studied in ref. 19. The steady state for arbitrary  $p$  and overall densities is obtained by factorizing the weights of the configurations into clusters of length two;<sup>(17)</sup> the strong short range correlations persist. The steady state for the generalization of the model where each particle has its own hopping probability has also been solved.<sup>(20)</sup>

Tilstra and Ernst<sup>(21)</sup> studied the case of OBC and  $p = 1$ . They obtained results which they argue to be asymptotically (i.e., in the limit of large system size) correct. In ref. 11, the system was found to be exactly solvable on a special line in the phase diagram; from this special case and extensive Monte Carlo simulations, the phase diagram and formulae for the current and the bulk densities were conjectured.

We now discuss briefly the nature of our solution. For the random sequential model, the initial breakthrough was the observation that there exists a recursion relation relating steady state weights (unnormalized probabilities) for a system of size  $N$  to those for a system of size  $N - 1$ .<sup>(12)</sup> Equivalently, one may write the weights as matrix elements or traces of products of operators; requiring these operators to satisfy certain algebraic rules then implies that the weights satisfy the recursion relations.<sup>(13)</sup> The matrix product allows a more direct calculation of steady state correlation functions than the recursion relations.<sup>(12, 22, 14)</sup>

For the present model we have followed a similar line of attack. When we write the weights as operator products, however, we must require that the operators satisfy quartic algebraic relations, which relate a product of four operators to sums of products of three and two operators (see Section 2). This is in contrast with the quadratic relations found in previous works.<sup>(13, 23–31)</sup> For example, the recursion relation for the weight  $f_N(\dots 0100\dots)$  relates systems of size  $N$  to systems of size  $N - 1$  and  $N - 2$  in the following way:

$$f_N(\dots 0100\dots) = (1 - p) f_{N-1}(\dots 010\dots) + f_{N-1}(\dots 000\dots) + p f_{N-2}(\dots 00\dots) \quad (1.1)$$

We believe that the above method (recursion relations in the system size) should be of general interest as an analytic approach to probabilistic cellular automata,<sup>(32–34)</sup> for which there are notoriously few exact results (see Schadschneider in ref. 1).

We now summarize the content of the paper. In Section 2 we define the model and provide the algebraic rules for a matrix product solution. The next section gives the proof that they indeed describe the stationary state; the argument proceeds by looking at blocks of consecutive particles and holes. In Section 4 we show that the quartic algebraic rules may be reduced to quadratic rules by assuming the operators are two by two matrices whose elements are matrices, generally of infinite dimension, i.e., that the operators are rank four tensors.

The reduction to quadratic algebraic rules allows us to relate the parallel update model to the model with other discrete-time updating schemes. Specifically in Section 5 we show that the current and density profile for parallel update are simply related to those quantities for ordered sequential and sublattice parallel updating, although the relation between higher order correlation functions is more complicated. Thus, in solving exactly the parallel model in Sections 8–10, we also obtain new exact results and prove conjectures for the other discrete-time models, for which only the asymptotic current was previously known.

In Section 6 several explicit representations of the matrices are constructed. As a first application, we solve the case  $p = 1$  by means of  $4 \times 4$  matrices in Section 7. A detailed analysis is made of the two point correlation functions in order to highlight the oscillating decay of the correlation function which is a particular feature under parallel dynamics.

We then turn to the task of obtaining the exact solution for general  $p < 1$ . The current phase diagram is derived using generating function techniques in Section 8, and the asymptotic behavior of the density profiles in Section 9 again using generating function techniques. The relevant Tauberian theorems are presented in Appendix A. For finite systems, we calculate exact combinatorial expressions for the density profiles and two point functions (Section 10). Technical details of the computations are contained in Appendix B. By combining the results from the two preceding sections, we compute the asymptotic bulk densities in Section 11. A discussion closes the paper.

## 2. MODEL DEFINITION AND STEADY STATE RECURSION RELATIONS

In this paper we study the asymmetric exclusion process with parallel dynamics and open boundary conditions. We consider a one dimensional

lattice, with  $N$  sites labeled 1 through  $N$ . Each site  $i$  may be occupied by a particle, in which case a binary variable  $\tau_i$  satisfies  $\tau_i=1$ , or empty, in which case  $\tau_i=0$ . The  $n$ -tuple  $\tau=(\tau_1, \dots, \tau_N)$  specifies the configuration of the system. The dynamics is defined by requiring that at each time step three things happen: (i) *all* particles on sites  $1, \dots, N-1$  with an empty site in front of them attempt to hop forwards, succeeding with probability  $p$ ; (ii) if site 1 is empty then a particle attempts to enter the lattice there, succeeding with probability  $\alpha$ ; and (iii) if site  $N$  is occupied then the particle there attempts to exit the lattice, succeeding with probability  $\beta$ . All of these processes are stochastically independent. Note the particle-hole symmetry: the removal of a particle at the right end can be viewed as an injection of a hole, so the dynamics is invariant under the combined operations of interchange of  $i$  and  $N-i+1$ , interchange of particles and holes, and interchange of  $\alpha$  and  $\beta$ . For example, we have

$$\langle \tau_i \rangle_N(\alpha, \beta, p) = 1 - \langle \tau_{N+1-i} \rangle_N(\beta, \alpha, p) \quad (2.1)$$

For  $p, \alpha$ , and  $\beta$  nonzero, the configuration 1010... can be reached from any other, so the model, viewed as a finite state Markov chain, has a single irreducible component and hence a unique steady state,<sup>(35)</sup> which we denote by  $P_N$ ;  $P_N(\tau)$  is the probability of finding a system of size  $N$  in configuration  $\tau$  in the long time limit. In calculating  $P_N(\tau)$  it is convenient, as noted in earlier work on the random sequential model,<sup>(12, 22, 13, 14)</sup> first to define unnormalized weights  $f_N(\tau)$  and then to recover the probabilities via

$$P_N(\tau) = f_N(\tau)/Z_N \quad (2.2)$$

where

$$Z_N = \sum_{\tau} f_N(\tau) \quad (2.3)$$

the sum taken over all configurations of size  $N$ .

The idea is now to introduce a *matrix product ansatz* by writing

$$f_N(\tau) = \langle W | \left( \prod_{i=1}^N (1\tau_i E + \tau_i D) \right) | V \rangle \quad (2.4)$$

This is to be read as a product of operators  $E$  and  $D$  (an  $E$  for each empty site and a  $D$  for each occupied site) contracted with a vector  $|V\rangle$  and dual vector  $\langle W|$ , yielding a scalar steady state weight; for example, if  $N=6$  and  $\tau=010001$  then  $f_N(\tau) = \langle W | E D E E E D | V \rangle$ . This method originated in work on the random sequential model,<sup>(13)</sup> in which the operators were

represented by infinite dimensional matrices. Later papers generalizing the original idea have employed operators represented by finite dimensional matrices<sup>(36, 37)</sup> or higher rank infinite dimensional tensors;<sup>(38, 39)</sup> both these approaches will be used here.

The operators  $E$  and  $D$  and vectors  $|V\rangle$  and  $\langle W|$  are required to satisfy certain algebraic rules, listed below. We determined these rules by finding the steady state explicitly for small system sizes and then guessing. We prove in the next section that these rules do indeed imply that the steady state of the system is given by (2.2) and (2.4) for general  $N$ , and in Section 6 we construct an explicit representation, thus verifying that the rules are consistent.

The rules for the bulk are

$$EDEE = (1 - p) EDE + EEE + pEE \quad (2.5)$$

$$EDED = EDD + EED + pED \quad (2.6)$$

$$DDEE = (1 - p) DDE + (1 - p) DEE + p(1 - p) DE \quad (2.7)$$

$$DDED = DDD + (1 - p) DED + pDD \quad (2.8)$$

We also have rules involving three sites next to each boundary,

$$DDE |V\rangle = (1 - \beta) DD |V\rangle + (1 - p) DE |V\rangle + p(1 - \beta) D |V\rangle \quad (2.9)$$

$$EDE |V\rangle = (1 - \beta) ED |V\rangle + EE |V\rangle + pE |V\rangle \quad (2.10)$$

$$\langle W| DEE = (1 - \alpha) \langle W| EE + (1 - p) \langle W| DE + p(1 - \alpha) \langle W| E \quad (2.11)$$

$$\langle W| DED = (1 - \alpha) \langle W| ED + \langle W| DD + p \langle W| D \quad (2.12)$$

and two sites next to each boundary,

$$DD |V\rangle = \frac{p(1 - \beta)}{\beta} D |V\rangle \quad (2.13)$$

$$ED |V\rangle = \frac{p}{\beta} E |V\rangle \quad (2.14)$$

$$\langle W| EE = \frac{p(1 - \alpha)}{\alpha} \langle W| E \quad (2.15)$$

$$\langle W| ED = \frac{p}{\alpha} \langle W| D \quad (2.16)$$

These rules permit the computation of all  $f_N(\tau)$  (up to an overall constant, which must be assumed to be nonzero). However, a rather indirect argument which we now discuss, is required when  $N=1$  or  $N=2$ . It is convenient, and represents no loss of generality, to assume that

$$\langle W | V \rangle > 0 \quad (2.17)$$

We may simplify  $\langle W | ED | V \rangle$  using either (2.14) or (2.16); equating the results shows that  $\alpha \langle W | E | V \rangle = \beta \langle W | D | V \rangle$ , so that we may write

$$\langle W | D | V \rangle = \frac{p}{\beta} \gamma \langle W | V \rangle \quad (2.18)$$

$$\langle W | E | V \rangle = \frac{p}{\alpha} \gamma \langle W | V \rangle \quad (2.19)$$

for some constant  $\gamma$ . Similarly,  $\langle W | DED | V \rangle$  can be simplified using either (2.12) or (2.14), and this leads to

$$\langle W | DE | V \rangle = (1 - \beta) \langle W | D | V \rangle + (1 - \alpha) \langle W | E | V \rangle + p\gamma \langle W | V \rangle \quad (2.20)$$

Relations (2.5)–(2.20) allow the straightforward computation of all  $f_N(\tau)$ .

Equation (2.5), when inserted in (2.4), leads to the recursion relation (1.1). From the set of all the algebraic rules one may similarly construct a whole set of recursion relations which uniquely specifies the steady state weights. It is more convenient, however, to work directly with the operator product.

The algebra is not well defined when  $\alpha$  or  $\beta$  vanishes, and the model is not interesting when  $p=0$ . However, one may consider the random sequential limit discussed in the introduction,  $\alpha = p\tilde{\alpha}$ ,  $\beta = p\tilde{\beta}$ ,  $p \rightarrow 0$ . Assuming that the operators  $E$ ,  $D$  and vectors  $\langle W |$ ,  $|V \rangle$  of our algebra have limits  $\tilde{E}$ ,  $\tilde{D}$ ,  $\langle \tilde{W} |$  and  $|\tilde{V} \rangle$  under this scaling, and also assuming that  $\gamma$  has the limit  $\tilde{\gamma} = 1$  (as it is true for the representations we construct in Section 6) one finds that (2.5)–(2.16) and (2.18)–(2.20) are, in the limit, consequences of the quadratic algebra of ref. 13, which reads

$$\tilde{D}\tilde{E} = \tilde{D} + \tilde{E} \quad (2.21)$$

$$\tilde{D} |\tilde{V} \rangle = \frac{1}{\tilde{\beta}} |\tilde{V} \rangle \quad (2.22)$$

$$\langle \tilde{W} | \tilde{E} = \frac{1}{\tilde{\alpha}} \langle \tilde{W} | \quad (2.23)$$

**Remark 2.1.** The relations (2.5)–(2.8) can be used to obtain the steady state for our model with periodic boundary conditions (i.e., on a ring) if (2.4) is replaced by  $f_N(\tau) = \text{Tr}(\prod_{i=1}^N (1 - \tau_i) E + \tau_i D)$ . However, the algebra is not needed in this simple case, because the steady state is already known.<sup>(2)</sup>

### 3. PROOF OF STATIONARITY

In this section we show that the operator algebra (2.5)–(2.12) may be used to compute the stationary state of the ASEP with parallel dynamics. An elementary recursive argument shows that the weights  $f_N(\tau)$  defined by (2.4) and the constant  $\gamma$  introduced in (2.18) and (2.19) satisfy  $f_N(\tau)/\gamma > 0$  for  $N \geq 1$ , so that (2.2) defines a probability distribution on the set of all system configurations. We must show that this distribution is invariant under the dynamics. To avoid consideration of many special cases it is convenient to first rewrite the algebraic relations satisfied by  $D$ ,  $E$ ,  $|V\rangle$ , and  $\langle W|$  in more unified form.

Note first that relations (2.5)–(2.8), (2.9)–(2.12), and (2.20) all involve four factors (from among  $D$ ,  $E$ ,  $\langle W|$ , and  $|V\rangle$ ) on their left hand sides. These relations may be expressed by the single equation

$$XDEY = a(XDY) XDY + a(XEY) XEY + pa(XY) XY \quad (3.1)$$

Here  $X$  denotes either  $\langle W|$ ,  $D$ , or  $E$  and  $Y$  denotes  $D$ ,  $E$ , or  $|V\rangle$ . The coefficient  $a(S)$  of a term  $S$  ( $S = XDY$ ,  $XEY$ , or  $XY$ ) on the right hand side of (3.1) is determined only by  $S$  itself, not the relation under consideration:

$$a(S) = \begin{cases} 1 - p, & \text{if } S \text{ contains } DE \\ 1 - \alpha, & \text{if } S \text{ contains } \langle W| E \\ 1 - \beta, & \text{if } S \text{ contains } D |V\rangle \\ \gamma, & \text{if } S \text{ is } \langle W| V\rangle \\ 1, & \text{otherwise} \end{cases} \quad (3.2)$$

(Equation (3.2) represents a slight abuse of notation, since  $a(S)$  really depends on the *form* of  $S$  rather than on the value of  $S$ , which may be an operator, a vector, or a scalar; no confusion should arise.) We obtain another form of the relation (3.1) by first writing  $XDEY = (1 - p) XDEY + pXDEY$  and then using (3.1) in the second term:

$$\begin{aligned} XDEY &= a(XDEY) XDEY + pa(XDY) XDY \\ &\quad + pa(XEY) XEY + p^2a(XY) XY \end{aligned} \quad (3.3)$$



The relations (2.13)–(2.16) and (2.18)–(2.19) can be similarly unified:

$$XD|V\rangle = \frac{p}{\beta} a(X|V\rangle) X|V\rangle \quad (3.4)$$

$$\langle W|EY = \frac{p}{\alpha} a(\langle W|Y) \langle W|Y \quad (3.5)$$

Here, as in (3.1),  $X$  is  $\langle W|$ ,  $D$ , or  $E$  and  $Y$  is  $D$ ,  $E$ , or  $|V\rangle$ . A second form of the relations (3.4) and (3.5) is obtained as was (3.3), starting from  $XFY = (1 - \beta)XFY + \betaXFY$  for (3.4) and  $XFY = (1 - \alpha)XFY + \alphaXFY$  for (3.5):

$$XD|V\rangle = a(XD|V\rangle) XD|V\rangle + pa(X|V\rangle) X|V\rangle \quad (3.6)$$

$$\langle W|EY = a(\langle W|EY) \langle W|EY + pa(\langle W|Y) \langle W|Y \quad (3.7)$$

**Remark 3.1.** (i) The set of terms on the right side of (3.1) is obtained by omitting either or both of the middle two factors in  $XDEY$ , and the set on the right side of (3.3) by omitting neither, either, or both. Similarly, terms on the right side of (3.4) and (3.5) arise through the omission of one operator, and those in (3.6) and (3.7) through the omission of zero or one. (ii) In (3.3), (3.6), and (3.7) the power of  $p$  in each term is the number of operators omitted in obtaining that term; in (3.1), it is one less than that number.

We now write down the stationarity condition which the weights  $f_N(\tau)$  must satisfy. If we imagine for the moment that our lattice contains two extra boundary sites, 0 on the left and  $N + 1$  on the right, then there are  $N + 1$  bonds  $(i, i + 1)$  across which an exchange might occur during one step in the evolution; here by an “exchange” across  $(0, 1)$  or  $(N, N + 1)$  we mean the entry or exit, respectively, of a particle. Given a fixed configuration  $\tau$ , let us write  $\mathcal{A}(\tau)$  for the subset of these bonds across which an exchange can occur in  $\tau$  and  $\mathcal{B}(\tau)$  for the complementary subset of bonds across which an exchange might have occurred in arriving at  $\tau$  from some immediate predecessor; the bonds in  $\mathcal{A}(\tau)$  correspond to  $\langle W|E$ ,  $DE$ , or  $D|V\rangle$  in the formula (2.4) for  $f_N(\tau)$ , while those in  $\mathcal{B}(\tau)$  correspond to  $\langle W|D$ ,  $ED$ , or  $E|V\rangle$ . For  $\mathcal{C} \subset \mathcal{B}(\tau)$  write  $\tau^{\mathcal{C}}$  for the configuration obtained from  $\tau$  by making the exchanges corresponding to the bonds in  $\mathcal{C}$ ; the configurations  $\tau^{\mathcal{C}}$ ,  $\mathcal{C} \subset \mathcal{B}(\tau)$ , comprise all possible immediate predecessors of  $\tau$ . Then the stationarity condition has the form

$$f_N(\tau) = \sum_{\mathcal{C} \subset \mathcal{B}(\tau)} \pi(\mathcal{C}) f_N(\tau^{\mathcal{C}}) \quad (3.8)$$

Here for any subset  $\mathcal{C}$  of  $\mathcal{B}(\tau)$ ,  $\pi(\mathcal{C})$  is the probability that, in the configuration  $\tau^{\mathcal{C}}$ , precisely the set of exchanges  $\mathcal{C}$  (out of the set  $\mathcal{A}(\tau^{\mathcal{C}})$  of possible exchanges) were in fact made; thus  $\pi(\mathcal{C})$  is a product of the following factors:

- $\alpha$ , if  $(0, 1) \in \mathcal{C}$ ;  $(1 - \alpha)$ , if  $(0, 1) \in \mathcal{A}(\tau^{\mathcal{C}}) \setminus \mathcal{C}$ ;
- $p$ , for each  $(i, i+1) \in \mathcal{C}$ ;  $(1 - p)$ , for each  $(i, i+1) \in \mathcal{A}(\tau^{\mathcal{C}}) \setminus \mathcal{C}$  ( $1 \leq i \leq N-1$ );
- $\beta$ , if  $(N, N+1) \in \mathcal{C}$ ;  $(1 - \beta)$ , if  $(N, N+1) \in \mathcal{A}(\tau^{\mathcal{C}}) \setminus \mathcal{C}$ ;

with  $\mathcal{A}(\tau^{\mathcal{C}}) \setminus \mathcal{C}$  denoting the set of bonds belonging to  $\mathcal{A}(\tau^{\mathcal{C}})$  but not to  $\mathcal{C}$ .

To verify (3.8) we will use the relations above to transform each side to a common form. We need one more concept. The configuration  $\tau$  may be divided into blocks of successive zeros and ones; let  $\mathcal{E}(\tau)$  denote the set of such blocks and for  $\mathcal{F} \subset \mathcal{E}(\tau)$  let  $\tau_{\mathcal{F}}$  denote the configuration obtained from  $\tau$  by omitting one operator from each block in  $\mathcal{F}$ .

We first consider the left hand side of (3.8). In the expression (2.4) for  $f_N(\tau)$  we apply (3.3) to each factor  $DE$ , (3.6) to each  $D|V\rangle$ , and (3.7) to each  $\langle W|E$ , if these occur. The resulting sum over all ways of omitting zero, one, or two operators at each of these bonds (see Remark 3.1.i) is equivalent to a sum over all ways of omitting zero or one operators from each block in  $\tau$ , so that from (3.2) and Remark 3.1.ii we obtain

$$\sum_{\mathcal{F} \subset \mathcal{E}(\tau)} p^{|\mathcal{F}|} (1 - \alpha)^{x(\mathcal{F})} (1 - p)^{y(\mathcal{F})} (1 - \beta)^{z(\mathcal{F})} f_{N-|\mathcal{F}|}(\tau_{\mathcal{F}}) \quad (3.9)$$

where  $|\mathcal{F}|$  is the number of elements in  $\mathcal{F}$  and  $x(\mathcal{F})$ ,  $y(\mathcal{F})$ , and  $z(\mathcal{F})$  count respectively the number of factors  $\langle W|E$ ,  $DE$ , and  $D|V\rangle$  in the expression (2.4) for  $f(\tau_{\mathcal{F}})$ . In the special case  $f_{N-|\mathcal{F}|}(\tau_{\mathcal{F}}) = \langle W|V\rangle$  which can occur only if  $\tau = 101010\dots$  or  $\tau = 010101\dots$ , there will be an additional factor of  $\gamma$ . It is important here that the coefficients  $a(S)$  in (3.3), (3.6), and (3.7) depend only on  $S$ , so that the coefficients in (3.9) depend only on  $\mathcal{F}$  and in particular are independent of the order in which the relations (3.3), (3.6), and (3.7) are applied at different bonds. Similar comments apply to the expansions using (3.1), (3.4), and (3.7) which we will perform below.

Consider now a weight  $f_N(\tau^{\mathcal{C}})$  which occurs in the sum on the right hand side of (3.8). Each bond in  $\mathcal{C}$  corresponds in the expression (2.4) for this weight to a factor  $DE$ ,  $D|V\rangle$ , or  $\langle W|E$ ; we apply (3.1), (3.4), or (3.5) to these factors. The result will be of the form

$$f_N(\tau^{\mathcal{C}}) = \sum_{\mathcal{F}} p^{|\mathcal{F}| - |\mathcal{C}|} \lambda(\mathcal{F}) f_{N-|\mathcal{F}|}(\tau_{\mathcal{F}}) \quad (3.10)$$

Since each of the relations we are using involves the omission of one or two operators (see Remark 3.1.i), the  $\mathcal{F}$  occurring in (3.10) will be those which, for each bond in  $\mathcal{C}$ , contain one or both of the blocks abutting on this bond. The total number of operators omitted is  $|\mathcal{F}|$  and the number of times one of the relations was used is  $|\mathcal{C}|$ , so that the factor  $p^{|\mathcal{F}|-|\mathcal{C}|}$  is obtained directly from Remark 3.1.ii. The coefficient  $\lambda(\mathcal{F})$  is a product which contains a factor  $p/\beta$  if (3.4) was used (i.e., if  $(0, 1) \in \mathcal{C}$ ) and  $p/\alpha$  if (3.5) was used ( $(N, N+1) \in \mathcal{C}$ ). It also contains factors arising from (3.2): a factor of  $1-p$ ,  $1-\beta$ , or  $1-\alpha$  for each application of (3.1), (3.4), or (3.5) which preserves or generates a factor  $DE$ ,  $D|V\rangle$ , or  $\langle W|E$  respectively, and a factor  $\gamma$  if  $f_{N-|\mathcal{F}|}(\tau_{\mathcal{F}}) = \langle W|V\rangle$ .

When (3.10) is inserted into the right side of (3.8) the double sum over  $\mathcal{C}$  and  $\mathcal{F}$  becomes a sum over all subsets  $\mathcal{F} \subset \mathcal{E}(\tau)$ ,

$$\sum_{\mathcal{F} \subset \mathcal{E}(\tau)} \pi(\mathcal{C}) p^{|\mathcal{F}|-|\mathcal{C}|} \lambda(\mathcal{F}) f_{N-|\mathcal{F}|}(\tau_{\mathcal{F}}) \tag{3.11}$$

since, given any  $\mathcal{F}$ , one may identify the corresponding  $\mathcal{C}$  in (3.10) as the set of bonds in  $\mathcal{B}(\tau)$  for which no block belonging to  $\mathcal{F}$  abuts on  $\mathcal{C}$ . It is straightforward to complete the proof by verifying that  $p^{|\mathcal{F}|-|\mathcal{C}|} \pi(\mathcal{C}) \lambda(\mathcal{F})$  is precisely the coefficient of  $f_{N-|\mathcal{F}|}(\tau_{\mathcal{F}})$  in (3.9), and hence that (3.9) and (3.11) agree. In particular,  $\pi(\mathcal{C}) \lambda(\mathcal{F})$  contains a factor  $p^{|\mathcal{C}|}$ , that is, one factor of  $p$  for each bond in  $\mathcal{C}$ ; for internal bonds these factors are present in  $\pi(\mathcal{C})$ , while if  $(0, 1) \in \mathcal{C}$  then  $\pi(\mathcal{C})$  contains a factor  $\alpha$  and  $\lambda(\mathcal{F})$  a factor  $p/\alpha$  (the argument for  $(N, N+1) \in \mathcal{C}$  is similar). If (3.9) contains a factor  $(1-\alpha)$ , that is, if  $f(\tau_{\mathcal{F}})$  contains  $\langle W|E$ , then  $\pi(\mathcal{C})$  contains this factor if  $(0, 1) \notin \mathcal{C}$  and  $\lambda(\mathcal{F})$  if  $(0, 1) \in \mathcal{C}$ ; the factors of  $(1-p)$  and  $(1-\beta)$  in (3.9) are accounted for similarly.

#### 4. REDUCTION TO A QUADRATIC ALGEBRA

In this section we show that tile quartic algebraic rules (2.5)–(2.16) can be reduced to quadratic rules by making a convenient choice for the operators involved. The trick is to write

$$D = \begin{pmatrix} D_1 & 0 \\ D_2 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

where  $D_1$ ,  $D_2$ ,  $E_1$ , and  $E_2$  are matrices of arbitrary (in general infinite) dimension; that is,  $D$  and  $E$  are written as rank four tensors with two

indices of (possibly) infinite dimension and the other two indices of dimension two. Correspondingly, we write  $\langle W|$  and  $|V\rangle$  in the form

$$\langle W| = (\langle W_1|, \langle W_2|), \quad |V\rangle = \begin{pmatrix} |V_1\rangle \\ |V_2\rangle \end{pmatrix} \quad (4.2)$$

where  $\langle W_1|$ ,  $\langle W_2|$ ,  $|V_1\rangle$ , and  $|V_2\rangle$  are vectors of the same dimension as  $D_1$  and  $E_1$ . We will show that the operators and vectors so defined satisfy the algebra of Section 2 if  $D_1$ ,  $E_1$ ,  $\langle W_1|$ , and  $|V_1\rangle$  satisfy the quadratic relations

$$D_1 E_1 = (1-p)[D_1 + E_1 + p] \quad (4.3)$$

$$D_1 |V_1\rangle = \frac{p(1-\beta)}{\beta} |V_1\rangle, \quad \langle W_1| E_1 = \langle W_1| \frac{p(1-\alpha)}{\alpha} \quad (4.4)$$

and  $D_2$ ,  $E_2$ ,  $\langle W_2|$ , and  $|V_2\rangle$  satisfy

$$E_2 D_2 = p[D_1 + E_1 + p] \quad (4.5)$$

$$E_2 |V_2\rangle = p |V_1\rangle, \quad \langle W_2| D_2 = \langle W_1| p \quad (4.6)$$

We now verify that (4.1)–(4.6) imply (2.5)–(2.20). First, by substituting (4.1) into the bulk relations (2.5)–(2.8) one finds that to satisfy the latter equations it is sufficient that

$$\begin{aligned} & E_1 D_1 E_1 + E_2 D_2 E_1 \\ &= (1-p) E_1 D_1 + (1-p) E_2 D_2 + E_1 E_1 + p E_1 \end{aligned} \quad (4.7)$$

$$\begin{aligned} & E_1 D_1 E_1 D_1 + E_2 D_2 E_1 D_1 + E_1 D_1 E_2 D_2 + E_2 D_2 E_2 D_2 \\ &= E_1 D_1 D_1 + E_2 D_2 D_1 + E_1 E_1 D_1 + E_1 E_2 D_2 + p E_1 D_1 + p E_2 D_2 \end{aligned} \quad (4.8)$$

$$\begin{aligned} & D_1 E_1 \\ &= (1-p) E_1 + (1-p) D_1 + p(1-p) \end{aligned} \quad (4.9)$$

$$\begin{aligned} & D_1 E_1 D_1 + D_1 E_2 D_2 \\ &= (1-p) E_1 D_1 + (1-p) E_2 D_2 + D_1 D_1 + p D_1 \end{aligned} \quad (4.10)$$

These relations follow from (4.3) and (4.5). For example, the left hand side of (4.7) becomes

$$\begin{aligned} & (1-p) E_1 [D_1 + E_1 + p] + p [D_1 + E_1 + p] E_1 \\ &= (1-p) E_1 D_1 + p D_1 E_1 + E_1 E_1 + p E_1 \end{aligned} \quad (4.11)$$

which another use of (4.3) and (4.5) shows to be equal to the right hand side. Similarly, with (4.1) and (4.2), the relations (2.13)–(2.16) involving two sites next to the right hand boundary follow from

$$\begin{pmatrix} D_1 D_1 |V_1\rangle \\ D_2 D_1 |V_1\rangle \end{pmatrix} = \frac{p(1-\beta)}{\beta} \begin{pmatrix} D_1 |V_1\rangle \\ D_2 |V_1\rangle \end{pmatrix} \quad (4.12)$$

$$E_1 D_1 |V_1\rangle + E_2 D_2 |V_1\rangle = \frac{p}{\beta} E_1 |V_1\rangle + \frac{p}{\beta} E_2 |V_2\rangle \quad (4.13)$$

and these equations are in turn implied by (4.4) and (4.6). The conditions (2.9) and (2.10), involving three sites next to the right boundary, are obtained similarly, although the manipulations involved become rather tedious. Relations at the left boundary are obtained from symmetry considerations, completing the verification.

In the remainder of the paper, we will consider only representations of the quadratic algebra having the form of (4.1) and (4.2). As we now discuss, computations in this representation are simplified by the fact that all quantities of physical interest may be expressed in terms of  $D_1$ ,  $E_1$ ,  $\langle W_1 |$ , and  $|V_1\rangle$ . This also leads to connections with other updating procedures for the ASEP (see Section 5).

Let us first derive such an expression for the normalization constant  $Z_N$ , given by

$$Z_N = \langle W | C^N |V\rangle \quad (4.14)$$

where

$$C = D + E = \begin{pmatrix} C_1 & E_2 \\ D_2 & 0 \end{pmatrix} \quad (4.15)$$

and  $C_1 = D_1 + E_1$ . Now,

$$C^N = \begin{pmatrix} G(N) & G(N-1) E_2 \\ D_2 G(N-1) & D_2 G(N-2) E_2 \end{pmatrix} \quad (4.16)$$

where by convention  $G(-1) = 0$  and

$$G(N) = \sum_{n=0}^N K^{N-n} (-p)^n \quad (4.17)$$

for  $N \geq 1$ , with

$$K = (C_1 + p) \quad (4.18)$$

This may be proven by first checking the case  $N = 0$  ( $G(0) = \mathbb{1}$ ) and then verifying the recursion  $G(N + 1) = C_1 G(N) + E_2 D_2 G(N - 1) = (K - p) G(N) + pKG(N - 1)$ . Note that

$$G(N) + pG(N - 1) = K^N \quad (4.19)$$

From (4.16), (4.19), and the action (4.6) of  $E_2$  and  $D_2$  on the boundary vectors, we have

$$\langle W | C^n = (\langle W_1 | K^n, \langle W_1 | K^{n-1} E_2) \quad (4.20)$$

$$C^n |V\rangle = \begin{pmatrix} K^n |V_1\rangle \\ D_2 K^{n-1} |V_1\rangle \end{pmatrix} \quad (4.21)$$

which leads to

$$Z_N = z_N + pz_{N-1} \quad (4.22)$$

where

$$z_n = \langle W_1 | K^n |V_1\rangle \quad (4.23)$$

An important quantity in determining the phase diagram is the current  $J_N$ , which is the probability that a particle passes through a particular bond in a particular time step. It is given by any one of three equivalent expressions:

$$J_N = \alpha \langle (1 - \tau_1) \rangle = p \langle \tau_i (1 - \tau_{i+1}) \rangle = \beta \langle \tau_N \rangle \quad (4.24)$$

where  $1 < i < N$ ; the second of these may be written using the algebra as

$$J_N = p \frac{\langle W | C^{i-1} D E C^{N-i-1} |V\rangle}{Z_N} \quad (4.25)$$

Now (4.20) and (4.21) yield

$$\langle W | C^n D = (\langle W_1 | K^{n-1} [D_1 + p], 0) \quad (4.26)$$

$$E C^n |V\rangle = \begin{pmatrix} [E_1 + p] K^{n-1} |V_1\rangle \\ 0 \end{pmatrix} \quad (4.27)$$

and the algebraic rule (4.3) implies that

$$[D_1 + p][E_1 + p] = K \quad (4.28)$$

so that from (4.25), (4.22), and (4.23),

$$J_N = \frac{pz_{N-1}}{z_N + pz_{N-1}} \quad (4.29)$$

This expression again involves only the matrices  $E_1$  and  $D_1$  and vectors  $\langle W_1 |$  and  $|V_1\rangle$ .

We may similarly express the one-point correlation function or density profile,

$$\langle \tau_i \rangle_N = \frac{1}{Z_N} \langle W | C^{i-1} D C^{N-i} | V \rangle \quad (4.30)$$

in terms of  $E_1$ ,  $D_1$ ,  $\langle W_1 |$ , and  $|V_1\rangle$ :

$$\langle \tau_i \rangle_N = \frac{\langle W_1 | K^{i-1} (D_1 + p) K^{N-i} | V_1 \rangle}{z_N + pz_{N-1}} \quad (4.31)$$

where we have employed (4.26) and (4.21). Similar expressions are possible for higher correlation functions; for example, the two-point correlation function  $\langle \tau_i (1 - \tau_j) \rangle_N$  is given, from (4.16), (4.26) and (4.27) by

$$\begin{aligned} \langle \tau_i (1 - \tau_j) \rangle_N &= \frac{1}{Z_N} \langle W | C^{i-1} D C^{j-i-1} E C^{N-j} | V \rangle \\ &= \frac{1}{Z_N} \langle W_1 | K^{i-1} (D_1 + p) G(j-i-1) (E_1 + p) K^{N-j} | V_1 \rangle \end{aligned} \quad (4.32)$$

Because  $G(n)$  is an alternating sum, (4.32) is more complicated than the corresponding expression in the random sequential case; this reflects the fact that stronger correlations exist under parallel dynamics than under random sequential dynamics.

In Section 10 we will give a more explicit formulae for  $\langle \tau_i \rangle_N$ ; see (10.12).

## 5. RELATIONS WITH OTHER MODELS

The reduction to a quadratic algebra and the expressions for the current and correlation functions, derived in Section 4, lead to relations between the ASEP with parallel updating and the same model with certain other discrete-time updating procedures. The procedures in question can in fact be defined for more general site variables and for any local dynamical rules which assign to each configuration of a pair of sites at time  $t$  a new

configuration at time  $t + 1$  with some given probability, and similarly to each configuration on the leftmost or rightmost site a new configuration. We recall these procedures briefly; see ref. 11 for precise definitions. In the *ordered sequential* update sites are updated one at a time, starting at the right end of the system and proceeding sequentially to the left end (backward ordered), or vice versa (forward ordered). In the *sublattice parallel* update, all site pairs  $i, i + 1$  with  $i$  even are updated at one time step and all such pairs with  $i$  odd at the next time step. Let us denote these procedures by the symbols  $T_{\leftarrow}$ ,  $T_{\rightarrow}$  and  $T_{\text{sp}}$  (more precisely  $T_{\leftarrow}$ ,  $T_{\rightarrow}$  and  $T_{\text{sp}}$  are the transfer matrices for the different procedures, however since in this discussion we do not require any properties of the transfer matrices, we do not give detailed definitions).

It can be shown<sup>(40)</sup> using the matrix product formalism that in general the procedures  $T_{\leftarrow}$ ,  $T_{\rightarrow}$  and  $T_{\text{sp}}$  lead to stationary states which may be regarded as physically equivalent. In particular, the current is independent of the update procedure; we write  $J^{\#}$  for this common value:

$$J_N^{\#} = J_N^{\leftarrow} = J_N^{\rightarrow} = J_N^{\text{sp}} \quad (5.1)$$

The density profiles are also closely related (we use the notation of the ASEP, but a corresponding result holds in general<sup>(40)</sup>):

$$\langle \tau_i \rangle_N^{\text{sp}} = \begin{cases} \langle \tau_i \rangle_N^{\leftarrow}, & i \text{ even,} \\ \langle \tau_i \rangle_N^{\rightarrow}, & i \text{ odd,} \end{cases} \quad \langle \tau_i \rangle_N^{\text{sp}*} = \begin{cases} \langle \tau_i \rangle_N^{\leftarrow}, & i \text{ odd} \\ \langle \tau_i \rangle_N^{\rightarrow}, & i \text{ even} \end{cases} \quad (5.2)$$

where  $\langle \tau_i \rangle_N^{\text{sp}*}$  is the density for  $T_{\text{sp}}$  after the first (even) sublattice has been updated. Similar relations hold for some higher order correlation functions.

More is known in the special case of the ASEP, to which we limit our discussion in what follows, based on simpler realizations of the matrix product ansatz in that case.<sup>(15, 10, 16, 11)</sup> In particular there is another relation between density profiles,

$$\langle \tau_i \rangle_N^{\rightarrow} = \langle \tau_i \rangle_N^{\leftarrow} - J_N^{\#} \quad (5.3)$$

which, with (5.2), shows that any one of  $\langle \tau_i \rangle_N^{\leftarrow}$ ,  $\langle \tau_i \rangle_N^{\rightarrow}$ , and  $\langle \tau_i \rangle_N^{\text{sp}}$  determines the others. Moreover, the asymptotic current  $\lim_{N \rightarrow \infty} J_N^{\#}$  (and therefore the phase diagram) is known,<sup>(16)</sup> and so are the density profiles and correlation functions in the case  $p = 1$ .<sup>(15, 10)</sup>

A fully parallel updating procedure is not naturally defined for arbitrary local dynamical rules, due to the possibility of conflict when the rules are applied simultaneously to pairs of overlapping sites, but for the ASEP such a fully parallel procedure, which we will denote by  $T_{\parallel}$ , has been defined in Section 2. We now want to relate this model to the ASEP



with update procedures  $T_{\rightarrow}$ ,  $T_{\leftarrow}$ , and  $T_{\text{sp}}$ , considered above; for the ASEP with parallel update we work with the reduced version of the algebra established in Section 4, in which everything is expressed in terms of the matrices  $E_1$  and  $D_1$  and vectors  $\langle W_1 |$  and  $|V_1\rangle$ , which satisfy the algebraic relations (4.3, 4.4). The idea is to introduce new operators  $e$  and  $d$  by

$$d = D_1, \quad e = E_1 + p \quad (5.4)$$

so that  $K = E_1 + D_1 + p = e + d$ . From (4.3, 4.4), the new matrices satisfy

$$de = d + (1 - p)e \quad (5.5)$$

and

$$d|V_1\rangle = \frac{p(1-\beta)}{\beta}|V_1\rangle, \quad \langle W_1|e = \langle W_1|\frac{p}{\alpha} \quad (5.6)$$

Surprisingly, these equations are precisely the algebraic relations<sup>(10)</sup> for the matrix product solution of the ASEP with forward updating, that is, the steady state weight for the configuration  $\tau$  with updating  $T_{\rightarrow}$  is given by

$$\langle W_1 | \left( \prod_{i=1}^N (1 - \tau_i) e + \tau_i d \right) | V_1 \rangle \quad (5.7)$$

Writing  $c = e + d$  we see that the normalizing factor for  $T_{\rightarrow}$  is

$$\langle W_1 | c^N | V_1 \rangle = \langle W_1 | K^N | V_1 \rangle = z_N \quad (5.8)$$

a constant already introduced in (4.23), and the current is

$$J_N^{\#} = \alpha \frac{\langle W_1 | e c^{N-1} | V_1 \rangle}{z_N} = p \frac{z_{N-1}}{z_N} \quad (5.9)$$

The density profile is given by

$$\langle \tau_i \rangle \rightarrow = \frac{\langle W_1 | c^{i-1} d c^{N-i} | V_1 \rangle}{z_N} = \frac{\langle W_1 | K^{i-1} D_1 K^{N-i} | V_1 \rangle}{\langle W_1 | K^N | V_1 \rangle} \quad (5.10)$$

These formulae imply certain simple relations between physical quantities in the parallel and the ordered sequential ASEP. From (5.9) and the formula (4.29) for the current in the ASEP with update  $T_{\parallel}$ ,  $J_N = p z_{N-1} / (z_N + p z_{N-1})$ , we have

$$J_N = \frac{J_N^{\#}}{1 + J_N^{\#}} \quad (5.11)$$

In particular,  $J_N < J_N^\#$ , a result which is intuitively clear. Similarly, comparing (5.10) with the formula (4.31) for the density profile for the parallel ASEP leads to

$$\langle \tau_i \rangle_N = \frac{\langle \tau_i \rangle_N^{\rightarrow} + J_N^\#}{1 + J_N^\#} = \frac{\langle \tau_i \rangle_N^{\leftarrow}}{1 + J_N^\#} \quad (5.12)$$

where we have also used (5.3). It is also possible to derive similar formulae for the higher correlation functions although these are not so simple. For example, the two-point function (4.32) for  $T_{||}$  can be written as an alternating sum over two-point functions of  $T_{\leftarrow}$ :

$$\begin{aligned} & \langle \tau_i(1 - \tau_j) \rangle_N \\ &= \frac{1}{Z_N} \sum_{n=0}^{j-i-1} (-p)^n [\langle \tau_i(1 - \tau_{j-n}) \rangle_{N-n}^{z_{N-n}} - p \langle \tau_i \rangle_{N-n-1}^{z_{N-n-1}}] \end{aligned} \quad (5.13)$$

although this formula appears to be too complicated to be of practical use in obtaining  $\langle \tau_i(1 - \tau_j) \rangle_N$  from  $\langle \tau_i(1 - \tau_j) \rangle_N^{\rightarrow}$  or vice versa. We emphasize that the formulae (5.11), (5.12), and (5.13) hold for all  $\alpha$ ,  $\beta$ , and  $p$ . Of course, using the formulae established earlier in this section, we obtain relations for the one- and two-point functions with updates  $T_{\leftarrow}$  and  $T_{\text{sp}}$ .

The relation (5.11) for the currents and the relation for the bulk densities which follows from (5.12) were already conjectured in ref. 11 (see Table 1 in that reference).

These relations between different models have various consequences. On the one hand, established results for  $T_{\leftarrow}$ ,  $T_{\rightarrow}$  and  $T_{\text{sp}}$  serve as a check for the results we will derive later; this applies to the asymptotic values of the currents and hence to the phase diagram which we derive in Section 8, and to some of the results in the special case  $p = 1$ .

On the other hand, and more importantly, most of the results that we will derive in later sections—explicit representations of the algebra, some of the detailed results for the case  $p = 1$ , asymptotics of density profiles, and finite volume formulae for current and density profiles—apply to all of the update procedures  $T_{||}$ ,  $T_{\leftarrow}$ ,  $T_{\rightarrow}$ , and  $T_{\text{sp}}$ . We will state these in terms of  $T_{||}$ , since that procedure is the main focus of this paper, but results for other updates are easily obtained via the formulae of this section.

Finally, we want to point out a further connection to another known model, the ASEP with random sequential update on a ring with a second class particle.<sup>(25, 26)</sup> In the ASEP with two species of particles a first class particle hops onto a vacant site to its right with rate one and exchanges

positions with a second class particle to its right with rate  $r$ , and a second class particle hops onto a vacant site to its right with rate  $s$ . Let us summarize some known results for this model. The stationary state on a ring can be written as a matrix product state;<sup>(25)</sup> the corresponding matrix algebra is given by (2.21)–(2.23), where now the matrix for first class particles is  $\tilde{D}$ , the matrix for holes is  $\tilde{E}$ , the matrix for the second class particle is  $|\tilde{V}\rangle\langle\tilde{W}|$ , and  $\tilde{\alpha}=r$ ,  $\tilde{\beta}=s$ . Since we are considering a closed system, it is convenient to work in a grand canonical ensemble<sup>(25)</sup> with fugacity  $x$  for first class particles. When the system contains just one second class particle, the density of first class particles  $i$  sites ahead of the second class particle in a ring of  $N$  sites is given by

$$\langle\tau_i\rangle_N^{\text{scp}} = x \frac{\langle\tilde{W}|\tilde{C}^{i-1}\tilde{D}\tilde{C}^{N-i-1}|\tilde{V}\rangle}{\langle\tilde{W}|\tilde{C}^{N-1}|\tilde{V}\rangle} \quad (5.14)$$

where  $\tilde{C} = x\tilde{D} + \tilde{E}$ , with  $x$  the fugacity.

Now we can map the algebra (5.5), (5.6) onto that of (2.21)–(2.23) by defining

$$(1-p)\tilde{D} = d, \quad \tilde{E} = e \quad (5.15)$$

and

$$\tilde{\alpha} = \frac{p}{\alpha}, \quad \tilde{\beta} = \frac{\beta(1-p)}{p(1-\beta)} \quad (5.16)$$

Then  $c = e + d$  is equal to  $\tilde{C}|_{x=1-p}$ , so that one can obtain the density profile for the ASEP with  $T_{\rightarrow}$ , and hence for the ASEP with  $T_{||}$ , from the grand-canonical density profile (5.14) seen from the second class particle: comparing (5.10) and (5.14) yields

$$(1-p)\langle\tau_i\rangle_N^{\rightarrow} |_{\alpha, \beta, p} = \langle\tau_i\rangle_{N+1}^{\text{scp}} |_{r=\alpha/p, s=\beta(1-p)/p(1-\beta), x=1-p} \quad (5.17)$$

Other quantities in the models can be related similarly. When  $r=s=1$ ,  $\langle\tau_i\rangle_N^{\text{scp}}$  is known exactly for both finite and infinite systems,<sup>(25)</sup> yielding  $\langle\tau_i\rangle_N^{\rightarrow}$  for the case  $\alpha=\beta=p$ . The single second-class particle model for general  $r$  and  $s$  has been studied<sup>(26)</sup> in the canonical ensemble of a fixed number of particles on the ring; to the extent to which the canonical and grand canonical ensembles are equivalent in the large  $N$  limit, these results should correspond with our results for the model with parallel update. We checked that indeed the phase diagram we will derive in Section 8 translates to the correct phase diagram for the model of ref. 26.

## 6. REPRESENTATIONS OF THE QUADRATIC ALGEBRA

In this section we discuss several explicit representations of the quadratic algebra (4.3), (4.4). The situation is like that for other, similar matrix product algebras: finite dimensional representations exist for a few special parameter values, and some representation, typically infinite dimensional, exists for all values.

In ref. 11 it was observed that for parameter values on the line

$$(1 - \alpha)(1 - \beta) = 1 - p \quad (6.1)$$

the weight of a configuration in the stationary state could be written as a product of factors corresponding to clusters of length two. Thus, we expect a simplification of our algebraic relations (4.3)–(4.6) along (6.1). Indeed, for this special case the matrices  $E_1$ ,  $D_1$ ,  $E_2$ ,  $D_2$  and the vectors  $\langle W_1|$ ,  $\langle W_2|$ ,  $|V_1\rangle$ ,  $|V_2\rangle$  can be chosen to be scalars:

$$D_1 = p \frac{1 - \beta}{\beta}, \quad E_1 = p \frac{1 - \alpha}{\alpha} \quad (6.2)$$

$$D_2 = E_2 = pv \quad (6.3)$$

$$|V_1\rangle = \langle W_1| = v \quad (6.4)$$

$$|V_2\rangle = \langle W_2| = 1 \quad (6.5)$$

where  $v = \sqrt{p/\alpha\beta}$ , so that  $E$  and  $D$  are  $2 \times 2$  matrices.

In Section 7 we will treat the case  $p = 1$  (with  $\alpha$  and  $\beta$  arbitrary), in which one can choose a two dimensional representation of the  $E_1$ ,  $D_1$  algebra. Here, the bulk dynamics is completely deterministic. However, the physics is still interesting,<sup>(21)</sup> since  $\alpha$  and  $\beta$  can induce different phases. The algebra is sufficiently simple that we can derive explicit formulae for quantities such as the fluctuations in the number of particles in the finite system.

Finite dimensional representations of the matrix product algebra (4.3), (4.4) exist only along the line (6.1) and when  $p = 1$ . This can be seen by using the mapping (5.15), (5.16) in Section 5, because it has been proven<sup>(13)</sup> that the matrices in (2.21)–(2.23) have to be infinite-dimensional except in the case  $\tilde{\alpha} + \tilde{\beta} = 1$ , which is by (5.16) equivalent to condition (6.1). Note that the mapping (5.15) and (5.16) is not well-defined for  $p = 1$ ; however, in this case we know that there is a finite dimensional representation, because we construct it (see Section 7).

Let us now turn to the case of general  $p$ ,  $\alpha$ ,  $\beta$ , in which the representations must be infinite dimensional. Such representations are of use both as a calculational tool and also to demonstrate that non-trivial representations

of the algebra actually do exist. It can be shown by direct calculation that the following expressions satisfy (4.3)–(4.6):

$$\langle W_1 | = (1, 0, 0 \dots), \quad |V_1\rangle = (1, 0, 0 \dots)^t \tag{6.6}$$

$$D_1 = \begin{pmatrix} p(1-\beta)/\beta & b & 0 & 0 & \cdot & \cdot \\ 0 & (1-p) & (1-p)^{1/2} & 0 & & \\ 0 & 0 & (1-p) & (1-p)^{1/2} & & \\ 0 & 0 & 0 & (1-p) & \cdot & \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \end{pmatrix} \tag{6.7}$$

$$E_1 = \begin{pmatrix} p(1-\alpha)/\alpha & 0 & 0 & 0 & \cdot & \cdot \\ b & (1-p) & 0 & 0 & & \\ 0 & (1-p)^{1/2} & (1-p) & 0 & & \\ 0 & 0 & (1-p)^{1/2} & (1-p) & & \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \end{pmatrix} \tag{6.8}$$

where

$$b^2 = \frac{p}{\alpha\beta} [(1-p) - (1-\alpha)(1-\beta)] \tag{6.9}$$

and

$$E_2 = gD_1, \quad D_2 = gE_1 \tag{6.10}$$

$$|V_2\rangle = \beta/(g(1-\beta)) |V_1\rangle, \quad \langle W_2 | = \langle W_1 | \alpha/(g(1-\alpha)) \tag{6.11}$$

where  $g = \sqrt{p/(1-p)}$ . These representations are not defined when  $p = 1$ , however, as explained above, we will treat this case in Section 7. Note that for  $(1-\alpha)(1-\beta) = 1-p$  we have  $b^2 = 0$ , so that the  $(1, 1)$  elements of the matrices decouple from the rest and we are left with  $2 \times 2$  operators  $D, E$  (see also 4.2). This proves a conjecture of ref. 11 for the structure of the steady state. In the limit  $p \rightarrow 0$  one recovers the representations (36) and (37) of ref. 13.

### 7. THE CASE $p = 1$

In the case  $p = 1$  the hopping of particles in the bulk is deterministic; the only source of randomness comes from the parameters  $\alpha$  and  $\beta$ . We

shall see that  $\alpha$  and  $\beta$  can induce a high density phase ( $\alpha \geq \beta$ ) and a low density phase ( $\beta \geq \alpha$ ). For  $\alpha = \beta$ , these two phases coexist (see below). Note that the recursion relation following from the algebraic rule (2.7) implies that the weight for configurations containing a particle-particle-hole-hole string is exactly zero. This is also immediately evident from the dynamical rules, since these substrings can never be created (they are ‘‘Gardens of Eden’’ that can never be entered once left).

One can choose  $2 \times 2$  representations for the operators  $E_1$  and  $D_1$ , so that  $E$  and  $D$  are  $4 \times 4$  matrices. In particular, one can verify that the following explicit representations satisfy (4.3)–(4.6):

$$\langle W_1 | = (1, 0), \quad |V_1\rangle = (1, 0)^t \quad (7.1)$$

$$D_1 = \begin{pmatrix} (1-\beta)/\beta & -c \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} (1-\alpha)/\alpha & 0 \\ c & 0 \end{pmatrix} \quad (7.2)$$

$$\langle W_2 | = (1/a, 0), \quad |V_2\rangle = (1/a, 0)^t \quad (7.3)$$

$$E_2 = \begin{pmatrix} a & -c \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \quad (7.4)$$

where

$$c = \sqrt{\frac{(1-\alpha)(1-\beta)}{\alpha\beta}}, \quad a = \sqrt{\frac{1}{\alpha\beta}} \quad (7.5)$$

In order to compute expectation values, it is convenient to diagonalize  $K$  (see (4.18)). It turns out that this can be done for  $\alpha \neq \beta$ ; we discuss this case first, but omit details of the diagonalization. However, note that the eigenvalues of  $K$  are  $1/\alpha$  and  $1/\beta$ .

Using the diagonalization, it is straightforward to compute  $z_N = \langle W_1 | K^N | V_1 \rangle$  by writing  $|V_1\rangle$  as a superposition of eigenvectors of  $K$  and one obtains

$$z_N = \frac{(1-\beta)\alpha\beta^{-N} - (1-\alpha)\beta\alpha^{-N}}{\alpha - \beta} \quad (7.6)$$

Equation (4.22) then leads to

$$Z_N = \frac{(1-\beta^2)\alpha\beta^{-N} - (1-\alpha^2)\beta\alpha^{-N}}{\alpha - \beta} \quad (7.7)$$

Thus, the current  $J_N$  follows from (4.29):

$$J_N = \alpha\beta \frac{(1-\beta)\alpha^N - (1-\alpha)\beta^N}{(1-\beta^2)\alpha^{N+1} - (1-\alpha^2)\beta^{N+1}} \quad (7.8)$$

From this equation, it is clear that there are only two phases (for  $\alpha \neq \beta$ ), a high density phase when  $\alpha > \beta$  and a low density phase when  $\beta > \alpha$ . As  $N \rightarrow \infty$ ,  $J_N$  approaches  $\beta/(1+\beta)$  or  $\alpha/(1+\alpha)$ , respectively. These results were conjectured in refs. 11 and 21.

With the diagonalization and (4.31) it is straightforward to work out a formula for the density profile  $\langle \tau_i \rangle_N(\alpha, \beta)$  ( $i = 1, \dots, N$ ):

$$\langle \tau_i \rangle_N = \frac{\alpha^{N+1}(1-\beta) - \beta^{N+1}\alpha(1-\alpha) - (\beta/\alpha)^i(1-\alpha)(1-\beta)\alpha^{N+1}}{(1-\beta^2)\alpha^{N+1} - (1-\alpha^2)\beta^{N+1}} \quad (7.9)$$

It follows that in the limit  $N \rightarrow \infty$  the density profile in the high density phase  $\alpha > \beta$  is given by

$$\langle \tau_i \rangle = \frac{1}{1+\beta} - \frac{1-\alpha}{1+\beta} (\beta/\alpha)^i \quad (7.10)$$

This implies that the density is constant for  $i \gg 1$  (in particular, in the bulk and near the right boundary), and decays exponentially near the left boundary (on a scale given by  $\log(\alpha/\beta)$ ) towards the bulk density:

$$\langle \tau \rangle_{\text{bulk}} = \frac{1}{1+\beta} \quad (7.11)$$

The corresponding formulae for the the low density phase  $\beta > \alpha$  can be obtained easily directly from (7.9) or via the particle hole symmetry (2.1). One obtains

$$\langle \tau \rangle_{\text{bulk}} = \frac{\alpha}{1+\alpha} \quad (7.12)$$

and an exponentially growing density profile near the right boundary. These values for the bulk densities are in agreement with conjectures in refs. 11 and 21.

The two-point correlation function (for  $\alpha \neq \beta$ ) can be computed via (4.32) and is given by the following expression:

$$\begin{aligned}
\langle \tau_i \tau_j \rangle &= f(\alpha, \beta, N)^{-1} \times \{ \beta^{N+1} \alpha^2 (\alpha - 1) (1 + \beta) + \alpha^{N+1} (1 + \alpha) (1 - \beta) \\
&\quad + \beta^{N+1} \alpha (\alpha - 1) (1 + \beta) (-\alpha)^{j-i} + \alpha^{N+1} (1 + \alpha) (\beta - 1) (-\beta)^{1+j-i} \\
&\quad + \alpha^{N+1} (\alpha - 1) (1 + \alpha) (1 - \beta) (\beta/\alpha)^i \\
&\quad + \alpha^{N+1} (\alpha - 1) (\beta - 1) (\alpha - \beta) (-\beta)^j (-\alpha)^i \\
&\quad + \alpha^{N+1} (\alpha - 1) (1 - \beta) (1 + \beta) \alpha (\beta/\alpha)^j \}
\end{aligned} \tag{7.13}$$

with  $i \leq j$  and

$$f(\alpha, \beta, N) = (1 + \alpha)(1 + \beta) [(1 - \beta^2) \alpha^{N+1} - (1 - \alpha^2) \beta^{N+1}] \tag{7.14}$$

From this one can derive asymptotic expressions; for example, in the high density phase ( $\alpha > \beta$ ) the bulk two-point correlation function ( $j = i + r$ ,  $1 \ll i \leq i + r$ ) is

$$\langle \tau_i \tau_{i+r} \rangle_{\text{bulk}} = \frac{1 + \beta(-\beta)^r}{(1 + \beta)^2} \tag{7.15}$$

Note the oscillating nature of (7.15), which can be interpreted as a particle-hole attraction which is created by the simultaneous updating (see introduction). The corresponding truncated correlation function,

$$g(i, j) = \langle \tau_i \tau_j \rangle - \langle \tau_i \rangle \langle \tau_j \rangle = \frac{\beta(-\beta)^{j-i}}{(1 + \beta)^2} \tag{7.16}$$

decays for  $\beta \neq 1$  exponentially to zero on a scale  $1/|\ln \beta|$ .

Let us now turn to the case  $\alpha = \beta$ . In that case,  $K$  cannot be diagonalized, but there exists a similarity transformation which reduces  $K$  to Jordan normal form and makes all the necessary calculations straightforward. We just list the results:

$$z_N = \frac{1}{\alpha^N} [N(1 - \alpha) + 1] \tag{7.17}$$

$$Z_N = \frac{1}{\alpha^N} [N(1 - \alpha^2) + \alpha^2 + 1] \tag{7.18}$$

$$J_N = \alpha \frac{N(1 - \alpha) + \alpha}{N(1 - \alpha^2) + 1 + \alpha^2} \tag{7.19}$$



and

$$\langle \tau_i \rangle_N = \frac{(1-\alpha)^2 i + \alpha N(1-\alpha) + \alpha}{N(1-\alpha^2) + 1 + \alpha^2} \quad (7.20)$$

which yields for large  $N$

$$\langle \tau_i \rangle_N = \frac{\alpha + (1-\alpha)(i/N)}{1+\alpha} + O(1/N) \quad (7.21)$$

The error term is small when  $N(1-\alpha) \gg 1$ . Again, these expressions coincide with the expected formulae.<sup>(11)</sup> The linear profile in (7.21) can be interpreted as arising from a uniform superposition of states with localized shocks.

The two point correlation function is now

$$\begin{aligned} \langle \tau_i \tau_j \rangle = t(\alpha, N)^{-1} \times \{ & (1-\alpha)(1-\alpha^2) i + \alpha(1-\alpha)(1-\alpha^2) j \\ & + N\alpha^2(1-\alpha^2) + 2\alpha^2 + (-\alpha)^{(j-i)} (\alpha(1-\alpha^2)(i-j) \\ & + N\alpha(1-\alpha^2) + \alpha(1+\alpha^2)) \} \end{aligned} \quad (7.22)$$

where  $i, j = 1, 2, \dots, L$ ,  $i \leq j$  and  $t(\alpha, N) = (1+\alpha)^2 [N(1-\alpha^2) + 1 + \alpha^2]$ . When  $\alpha = 1$  this reduces to

$$\langle \tau_i \tau_j \rangle_{\alpha=1} = \frac{1}{4} [1 + (-1)^{(j-i)}] \quad (7.23)$$

which is the expected result since then in the steady state only the two configurations in which there are no particle-particle or hole-hole pairs occur with non zero probability. The alternating structure of (7.23) does not show up in the density profile (which is flat here) because these two configurations have equal weights.

For  $\alpha \neq 1$ , the truncated correlation function  $g(i, j)$  simplifies for large  $N$ :

$$g(i, j) = \frac{1}{(1+\alpha)^2} \{ (i/N)(1 - (j/N))(1-\alpha)^2 + (-\alpha)^r \alpha(1 + (r/N)) \} + O(1/N) \quad (7.24)$$

where  $r = j - i$ . Let us briefly discuss the behavior of  $g(i, j)$  for fixed  $i$  and  $j = i, i + 1, \dots, N$ . The oscillating part of  $g(i, j)$ , which is not present with other updating schemes, decays exponentially with  $j$  on a scale  $1/|\ln \alpha|$ . Therefore, for sufficiently large  $j$ ,  $g(i, j)$  decays linearly to zero with slope  $-((1-\alpha)/(1+\alpha))^2 (i/N^2)$ . Figure 1 shows two examples of  $g(i, j)$  for a system of 100 sites. The strong oscillations present for  $\alpha = \beta = 0.9$  arise

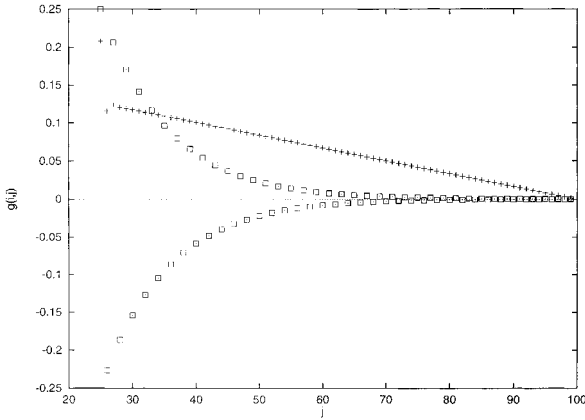


Fig. 1. The exact truncated correlation function  $g(i, j)$  in the case  $p=1$ , for  $i=25$  versus  $j$ . The system size is  $N=100$ . The oscillating curve (squares) was obtained for  $\alpha=\beta=0.9$ , the other curve (crosses) for  $\alpha=\beta=0.1$ .

because the density in the system is nearly at its maximum value of  $1/2$ , so that if a site is empty its two nearest neighbors are probably occupied. When  $\alpha=\beta=0.1$ , on the other hand, each site is for some typical configurations in a region of low density (if the shock is to its right) and for some in a region of high density (if the shock is to its left), so that the truncated correlations are positive.

We now turn to the calculation of the fluctuations in the number  $M$  of particles in the system, still considering the case  $\alpha=\beta$ . We write

$$\langle M \rangle = \sum_{i=1}^N \langle \tau_i \rangle \quad (7.25)$$

$$\Delta^2 = \langle M^2 \rangle - \langle M \rangle^2 = 2 \sum_{i < j}^N \langle \tau_i \tau_j \rangle + \langle M \rangle - \langle M \rangle^2 \quad (7.26)$$

and must sum up the expressions (7.20) and (7.22), respectively. The first summation is trivial:

$$\langle M \rangle = \frac{N^2(1-\alpha^2) + N(1+\alpha^2)}{2N(1-\alpha^2) + 2(1+\alpha^2)} \quad (7.27)$$

For  $\alpha=\beta=1$  we obtain the expected result  $\langle M \rangle_{\alpha=1} = N/2$ . The summation of (7.22) is more tedious. It is convenient to use  $\sum_{j, i < j}^N h(r) = \sum_{r=1}^{N-1} (N-r) h(r)$ , where  $h(r)$  is an arbitrary function of  $r$ . One is then left

with well-known sums of the form  $\sum_{r=1}^{N-1} r^k (-\alpha)^r$  ( $k = 0, 1, 2$ ). Altogether one obtains

$$\begin{aligned} \Delta^2(\alpha, N) = t(\alpha, N)^{-1} \times & \left\{ \frac{N^3}{3} [1 + \alpha - \alpha^3 - \alpha^4] - N^2 \alpha^2 \left[ \frac{1 - 3\alpha - \alpha^2 - \alpha^3}{1 + \alpha} \right] \right. \\ & - 2N\alpha^2 \left[ \frac{2 + \alpha + \alpha^2}{1 + \alpha} \right] + (1 - \alpha + \alpha^2) \left( \frac{2\alpha}{1 + \alpha} \right)^2 \\ & \left. - (-\alpha)^N \left( \frac{2\alpha}{1 + \alpha} \right)^2 [N(1 - \alpha^2) + 1 - \alpha + \alpha^2] \right\} + \langle M \rangle - \langle M \rangle^2 \end{aligned} \quad (7.28)$$

For  $\alpha = 1$  (7.28) yields  $1/4$  for  $N$  odd, and  $0$  for  $N$  even, as expected.

We now take  $N \gg 1$  and keep only the highest order in  $N$ . This gives

$$\Delta^2 = \frac{N^2 (1 - \alpha)^2}{12 (1 + \alpha)^2} \quad (7.29)$$

which can be rewritten, using (7.21), as

$$\Delta^2 = \frac{N^2}{12} (\rho_{\text{right}} - \rho_{\text{left}})^2 \quad (7.30)$$

where  $\rho_{\text{right}}$  ( $\rho_{\text{left}}$ ) is the asymptotic density at position  $N(1)$  given by (7.21). This is precisely the result which is to be expected when one considers the linear profile as a superposition of uniformly distributed random shock positions (step functions). The two-point correlation functions for the model with sublattice-parallel updating, obtained in refs. 9 and 15 also lead to (7.30), although the corrections to  $\Delta^2$  at lower order in  $N$  are different in both models.

## 8. DERIVATION OF THE PHASE DIAGRAM FOR GENERAL $p$

In this section we determine the asymptotic behavior, for all values of  $\alpha, \beta, p$ , of the quantity  $z_N = \langle W_1 | K^N | V_1 \rangle$  introduced in (4.23) and hence, through (4.29), of the current  $J_N$ ; the different possible asymptotic forms determine the distinct phases of the model. Our method is to study the generating function

$$\Theta_0(\lambda) \equiv \sum_{N=0}^{\infty} \lambda^N z_N \quad (8.1)$$

We will use the explicit representation (6.6)–(6.8) of the operators  $D_1$  and  $E_1$  and the vectors  $\langle W_1 |$  and  $|V_1\rangle$ , and will write  $|n\rangle$ ,  $n=0, 1, \dots$ , for the basis of the space on which  $D_1$  and  $E_1$  act, and  $\langle n|$  for the dual basis, so that  $|V_1\rangle = |0\rangle$  and  $\langle W_1| = \langle 0|$ . Note that  $z_0 = \langle W_1 | V_1 \rangle = 1$ .

From (6.7) and (6.8) it follows that  $K \equiv E_1 + D_1 + p$  has the form

$$K = \begin{pmatrix} c & b & 0 & 0 & \cdot & \cdot \\ b & 2-p & (1-p)^{1/2} & 0 & & \\ 0 & (1-p)^{1/2} & 2-p & (1-p)^{1/2} & & \\ 0 & 0 & (1-p)^{1/2} & 2-p & \cdot & \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \end{pmatrix}$$

where  $c = p(\alpha + \beta - \alpha\beta)/\alpha\beta$  and  $b$  is given by (6.9). From (8.2) we find

$$\langle 0 | K^{N+1} | 0 \rangle = c \langle 0 | K^N | 0 \rangle + b \langle 1 | K^N | 0 \rangle \quad (8.3)$$

$$\begin{aligned} \langle 1 | K^{N+1} | 0 \rangle &= b \langle 0 | K^N | 0 \rangle + (2-p) \langle 1 | K^N | 0 \rangle \\ &\quad + (1-p)^{1/2} \langle 2 | K^N | 0 \rangle \end{aligned} \quad (8.4)$$

and for  $n > 1$ ,

$$\begin{aligned} \langle n | K^{N+1} | 0 \rangle &= (1-p)^{1/2} \langle n-1 | K^N | 0 \rangle \\ &\quad + (2-p) \langle n | K^N | 0 \rangle + (1-p)^{1/2} \langle n+1 | K^N | 0 \rangle \end{aligned} \quad (8.5)$$

If we now define the generating functions

$$\Theta_n = \sum_{N=0}^{\infty} \lambda^N \langle n | K^N | 0 \rangle \quad (8.6)$$

we easily obtain from (8.3)–(8.5), using  $\langle n | 0 \rangle = \delta_{n,0}$ , that

$$(1 - c\lambda) \Theta_0 = b\lambda\Theta_1 + 1 \quad (8.7)$$

$$(1 - (2-p)\lambda) \Theta_1 = b\lambda\Theta_0 + (1-p)^{1/2} \lambda\Theta_2 \quad (8.8)$$

and for  $n > 1$ ,

$$(1 - (2-p)\lambda) \Theta_n = (1-p)^{1/2} \lambda\Theta_{n-1} + (1-p)^{1/2} \lambda\Theta_{n+1} \quad (8.9)$$

The solution of (8.9) is

$$\Theta_n = Au^n \quad \text{for } n > 0 \quad (8.10)$$

where  $A$  is a constant to be determined from (8.8) and

$$u = \frac{1 - \lambda(2 - p) - \sqrt{(1 + \lambda p)^2 - 4\lambda}}{2\lambda(1 - p)^{1/2}} \quad (8.11)$$

(the positive root is discarded being non-analytic at  $\lambda = 0$ ). Writing  $\Theta_1 = Au$  and  $\Theta_2 = Au^2$  in (8.7) and (8.8), and eliminating  $A$  from the resulting equations, yields

$$\Theta_0 = \frac{1}{1 - c\lambda - b^2\lambda u/(1 - p)^{1/2}} \quad (8.12)$$

From (8.12), (8.11), and some tedious algebra, we find that

$$\Theta_0(\lambda) = \frac{\alpha\beta[2(1 - p)(\alpha\beta - p^2\lambda) - \alpha\beta b^2(1 - p\lambda) - \alpha\beta b^2 \sqrt{(1 + p\lambda)^2 - 4\lambda}]}{2p^4(1 - \beta)(1 - \alpha)(\lambda - \lambda_{\text{hd}})(\lambda - \lambda_{\text{ld}})} \quad (8.13)$$

where

$$\lambda_{\text{ld}} = \frac{\alpha(p - \alpha)}{p^2(1 - \alpha)} \quad (8.14)$$

$$\lambda_{\text{hd}} = \frac{\beta(p - \beta)}{p^2(1 - \beta)} \quad (8.15)$$

Equation (8.13) shows that  $\Theta_0$  has square root singularities at the two points

$$\lambda_{\text{mc}}^{\pm} = \frac{2 - p \pm 2\sqrt{1 - p}}{p^2} \quad (8.16)$$

which, if we assume that the parameters  $\alpha$ ,  $\beta$ , and  $p$  lie in the relevant range  $0 \leq \alpha, \beta, p \leq 1$ , are on the positive real axis. Thus  $\Theta_0$  is naturally double valued and has single valued determinations (branches) on each of two *sheets*—copies of the complex plane cut along the real axis between the two roots. We are primarily interested in the behavior on the *first sheet*, the plane on which, for  $\lambda$  small and real, the square root in (8.13) is positive and hence  $\Theta_0(0) = 1$  (see (8.1)).  $\Theta_0$  also has two simple poles, at  $\lambda_{\text{ld}}$  and  $\lambda_{\text{hd}}$ .

As discussed in Appendix A, the coefficients  $z_N$  in the power series (8.1) will grow as  $N^\gamma \lambda_0^N$ , where  $\lambda_0$  is the singularity of  $\Theta_0(\lambda)$  on the first

sheet nearest to the origin ( $\lambda_0 > 1$  always) and  $\gamma$  is determined by the nature of that singularity. Thus

$$\lim_{N \rightarrow \infty} \frac{z_N}{z_{N+1}} = \lambda_0 \quad (8.17)$$

and hence

$$\lim_{N \rightarrow \infty} J_N = \frac{p\lambda_0}{1 + p\lambda_0} \quad (8.18)$$

Three regions in the parameter space must be considered according to which of the three singularities is closest to the origin. As we shall see, the singularities  $\lambda_{\text{ld}}$  and  $\lambda_{\text{hd}}$  may or may not be present on the first sheet of the complex plane. For the parameter values where one or both do occur they are closer to the origin than  $\lambda_{\text{mc}}^-$ . It is convenient to introduce the quantity

$$q = q(p) = 1 - \sqrt{1-p} \quad (8.19)$$

in discussing the resulting phase diagram (thus  $2q - p = q^2$  and  $\lambda_{\text{mc}}^- = (q/p)^2$ ). The phase diagram is shown in Fig. 2.

(i) **The maximum current region:**  $q < \alpha$  and  $q < \beta$  (region C in Fig. 2). For these parameter values the numerator in (8.13) vanishes at  $\lambda_{\text{ld}}$

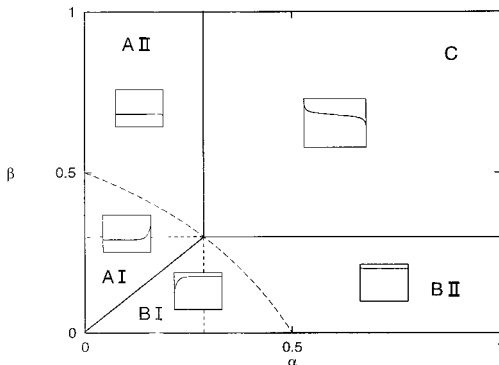


Fig. 2. (taken from ref. 11): Phase diagram for the ASEP with parallel (synchronous) update for  $p=0.5$ . C is the maximum current phase, A and B are the low and high density phase, respectively. The straight dashed lines are the boundaries between phase A I and A II (B I and B II). The curved dashed line is the line given by (6.1) and intersects the line  $\alpha = \beta$  at  $\alpha = \beta = 1 - \sqrt{1-p} = q$  (see Section 8). The inserts show typical density profiles in the various phases; note that the profile is qualitatively the same in region A I (B I) and in the portion of region A II (B II) below the curved dashed line.

and  $\lambda_{\text{hd}}$  when the square root is positive: the poles lie on the second sheet and the singularity closest to the origin in the first sheet is  $\lambda_{\text{mc}}^-$ . Then (A.7) implies that

$$z_N = \frac{\alpha^2 \beta^2 b^2 (1-p)^{1/4} \sqrt{\lambda_{\text{mc}}^-}}{2 \sqrt{\pi} (\alpha-q)^2 (\beta-q)^2 N \sqrt{N} (\lambda_{\text{mc}}^-)^N} + O(N^{-5/2} (\lambda_{\text{mc}}^-)^N) \quad (8.20)$$

and hence from (8.18),

$$\lim_{N \rightarrow \infty} J_N = \frac{p \lambda_{\text{mc}}^-}{1 + p \lambda_{\text{mc}}^-} = \frac{1 - \sqrt{1-p}}{2} \quad (8.21)$$

Note that the prefactor in (8.20) is singular at the boundaries  $\alpha=q$ ,  $\beta=q$  of the maximum current region; near these boundaries one needs larger values of  $N$  for the leading term in (8.20) to well approximate  $z_n$ .

(ii) **The low density region:**  $\alpha < \beta$  and  $\alpha < q$  (region A in Fig. 2). In this region the pole  $\lambda_{\text{id}}$  lies on the first sheet and is in fact the singularity of  $\Theta_0(\lambda)$  closest to the origin, and from (A.5),

$$z_N = \frac{\beta(p + \alpha^2 - 2\alpha)}{(p-\alpha)(\beta-\alpha)} \frac{1}{(\lambda_{\text{id}})^{-N}} + o(s^{-N}) \quad (8.22)$$

for some  $s > \lambda_{\text{id}}$ . Thus from (8.18),

$$\lim_{N \rightarrow \infty} J_N = \frac{p \lambda_{\text{id}}}{1 + p \lambda_{\text{id}}} = \frac{\alpha(p-\alpha)}{p-\alpha^2} \quad (8.23)$$

(iii) **The high density region:**  $\beta < \alpha$  and  $\beta < q$  (region B in Fig. 2). The asymptotic behavior of  $z_N$  here is obtained from (8.22) by interchanging  $\alpha$  and  $\beta$  and replacing  $\lambda_{\text{id}}$  by  $\lambda_{\text{hd}}$ . In particular,

$$\lim_{N \rightarrow \infty} J_N = \frac{\beta(p-\beta)}{p-\beta^2} \quad (8.24)$$

## 9. ASYMPTOTICS OF THE DENSITY FOR GENERAL $p$

In this section we calculate the behavior of the particle density near the left end of the system in the limit of infinite system size and for all values of  $\alpha$ ,  $\beta$  and  $p$ ; behavior near the right end can be recovered from the symmetry (2.1). For  $m, n \geq 0$  let

$$t_{m,n} = \lim_{N \rightarrow \infty} \frac{\langle W_1 | K^{N-m-n} D_1^m K^n | V_1 \rangle}{z_N} \quad (9.1)$$

and introduce the generating function

$$\Psi(x, y) = \sum_{m, n \geq 0} x^m y^n t_{m, n} \quad (9.2)$$

Our goal is to calculate  $\Psi_x(0, y)$ . For it follows from (4.31) and (8.17) that  $\rho_n$ , the density at the  $(n+1)$ st site to the left of the right boundary in the infinite volume limit (where  $n=0, 1, \dots$ ), is given by

$$\rho_n = \lim_{N \rightarrow \infty} \langle \tau_{N-n} \rangle_N = \lim_{N \rightarrow \infty} \frac{\langle W_1 | K^{N-1-n} (D_1 + p) K^n | V_1 \rangle}{z_N + p z_{N-1}} = \frac{t_{1, n} + p \lambda_0}{1 + p \lambda_0} \quad (9.3)$$

so that the generating function  $\Phi(y)$  for the  $\rho_n$  is

$$\begin{aligned} \Phi(y) &= \sum_{n \geq 0} y^n \rho_n = \frac{1}{1 + p \lambda_0} \sum_{n \geq 0} y^n (t_{1, n} + p \lambda_0) \\ &= \frac{1}{1 + p \lambda_0} \left( \Psi_x(0, y) + \frac{p \lambda_0}{1 - y} \right) \end{aligned} \quad (9.4)$$

Now  $t_{0, n} = 1$  for all  $n$ , so that  $\Psi(0, y) = (1 - y)^{-1}$ , and from (4.4) and (8.17),  $t_{m, 0} = (\lambda_0 p (1 - \beta) / \beta)^m$  for all  $m$ , so that  $\Psi(x, 0) = \beta / (\beta - x \lambda_0 p (1 - \beta))$ . From (4.3) it follows that  $D_1 K = D_1^2 + (1 - p) K + p D_1$  and this, together with (8.17), implies that for  $m, n \geq 1$ ,  $t_{m, n}$  satisfies the recursion

$$t_{m, n} = t_{m+1, n-1} + \lambda_0 (1 - p) t_{m-1, n} + \lambda_0 p t_{m, n-1} \quad (9.5)$$

Thus

$$\begin{aligned} \Psi(x, y) &= \sum_{m \geq 0} x^m t_{m, 0} + \sum_{n \geq 1} y^n t_{0, n} + \sum_{m, n \geq 1} x^m y^n t_{m, n} \\ &= \Psi(x, 0) + [\Psi(0, y) - 1] \\ &\quad + \sum_{m, n \geq 1} x^m y^n [t_{m+1, n-1} + \lambda_0 (1 - p) t_{m-1, n} + \lambda_0 p t_{m, n-1}] \\ &= \Psi(x, 0) + [\Psi(0, y) - 1] + \frac{y}{x} [\Psi(x, y) - \Psi(0, y) - x \Psi_x(0, y)] \\ &\quad + \lambda_0 (1 - p) x [\Psi(x, y) - \Psi(x, 0)] \\ &\quad + \lambda_0 p y [\Psi(x, y) - \Psi(0, y)] \end{aligned} \quad (9.6)$$



Multiplying (9.6) by  $-x$ , collecting terms, and using the relation  $\Psi(0, y) - 1 = y\Psi(0, y)$ , we see that  $\Psi$  satisfies the equation

$$D(x, y) \Psi(x, y) = A(x, y) + xy\Psi_x(0, y) \quad (9.7)$$

where

$$D(x, y) = \lambda_0(1-p)x^2 - (1-\lambda_0py)x + y \quad (9.8)$$

$$A(x, y) = -y[x(1-\lambda_0p) - 1] \Psi(0, y) - x[1 - \lambda_0(1-p)x] \Psi(x, 0) \quad (9.9)$$

Now the branch of the curve  $D(x, y) = 0$  given by  $x = \xi_-(y)$ , where

$$\xi_{\pm}(y) = \frac{1 - \lambda_0py \pm \sqrt{A(y)}}{2\lambda_0(1-p)} \quad (9.10)$$

with

$$A(y) = (1 + \lambda_0py)^2 - 4\lambda_0y \quad (9.11)$$

are the roots in  $x$  of  $D(x, y) = 0$ , passed through the origin. But  $\Psi(x, y)$  is analytic at the origin, so that (9.7) can hold only if the right hand side vanishes on this curve. This yields the desired equation for  $\Psi_x(0, y)$ :

$$\Psi_x(0, y) = -\frac{A(\xi_-(y), y)}{\xi_-(y)y} = \frac{\xi_-(y)(1-\lambda_0p) - 1}{\xi_-(y)(1-y)} + \frac{\beta[1 - \lambda_0(1-p)\xi_-(y)]}{y[\beta - \xi_-(y)\lambda_0p(1-\beta)]} \quad (9.12)$$

If we insert (9.12) into (9.4) and rationalize the resulting expression we obtain

$$\Phi(y) = \frac{1}{1+p\lambda_0} \left( \frac{2y-1+\lambda_0py}{2y(1-y)} - \frac{\sqrt{A(y)}}{2y(1-y)} + \frac{\beta(p-\beta)\sqrt{A(y)}}{2yp^2(1-\beta)(\lambda_{hd}-\lambda_0y)} + \frac{\beta((p-\beta)-\lambda_0py(2-\beta-p))}{2yp^2(1-\beta)(\lambda_{hd}-\lambda_0y)} \right) \quad (9.13)$$

We will always assume that the parameters in (9.13) lie in the physical region  $0 \leq \alpha, \beta, p \leq 1$ . Under this assumption the two roots of the equation  $A(y) = 0$  lie on the positive real axis, so that we may regard  $\Phi$  as defined on two sheets, as we did  $\Theta_0$  in the previous section. The first sheet corresponds, for  $y$  real and small, to  $\sqrt{A(y)} > 0$  in (9.13).

From (9.13) we see that the singular points of  $\Phi$  (which may coincide for some parameter values) are:

- A simple pole at  $y = 1$ .
- Two square root singularities  $y_{\pm}$ , the roots of the equation  $\Delta(y) = 0$ ; from (8.16),

$$y_{\pm} = \lambda_{\text{mc}}^{\pm} / \lambda_0 \quad (9.14)$$

Since  $\lambda_0 \leq \lambda_{\text{mc}}^-$ , these singularities satisfy  $1 \leq y_- \leq y_+$ .

- An apparent simple pole at

$$y_1 = \frac{\lambda_{\text{hd}}}{\lambda_0} = \frac{\beta(p - \beta)}{\lambda_0 p^2 (1 - \beta)} \quad (9.15)$$

However, the numerators of the third and fourth terms in (9.13) may be equal in magnitude and opposite in sign when  $y = y_1$ , cancelling this singularity; from (9.15) and (9.13) we find that this happens when

$$p(1 - \beta) - \beta(2 - \beta - p) = -p(1 - \beta) \sqrt{\Delta(y_1)} \quad (9.16)$$

A little algebra shows that the squares of the two sides in (9.16) are equal, so that (9.16) holds on the first sheet, and the pole at  $y_1$  is absent there, if  $p(1 - \beta) - \beta(2 - \beta - p) \leq 0$ , i.e., if  $\beta \geq q$ .

- An apparent simple pole at  $y = 0$ . From (9.4), however, it follows that  $\Phi$  is regular at the origin on the first sheet. This can also be seen directly from (9.13) using  $\sqrt{\Delta(0)} = 1$ .

We now analyze this generating function in the various regions of the phase plane of the system.

(i) **The maximum current region:**  $q < \alpha$  and  $q < \beta$  (region C in Fig. 2). In this region  $\lambda_0 = \lambda_{\text{mc}}^-$ , so that from (9.14) the square root singularity  $y_-$  coincides with the pole at  $y = 1$ ; thus  $\Delta(y)$  has a factor  $(1 - y)$  and from (9.11),

$$\Delta(y) = (1 - y)(1 - p^2(\lambda_{\text{mc}}^-)^2 y) \quad (9.17)$$

Since  $q < \beta$  there is no pole at  $y_1$  on the first sheet, and thus  $y = 1$  is the singularity closest to the origin and controls the asymptotics of the coefficients  $\rho_n$  of  $\Phi$ . We write  $\rho_n = \rho_n^{(1)} + \rho_n^{(2)} + \rho_n^{(3)}$ , where  $\rho_n^{(i)}$  is the contribution from the  $i$ th term in (9.13) (the fourth term is regular at  $y = 1$ ), and calculate the asymptotic form of each  $\rho_n^{(i)}$  in turn.

The first term in (9.13) is  $f_1(y)(1-y)^{-1}$ , where  $f_1(y) = (2y-1 + \lambda_{\text{mc}}^- py)/(2y(1+p\lambda_{\text{mc}}^-))$ . Thus, from (A.5) and (A.2),

$$\rho_n^{(1)} = f_1(1) + o(s^{-n}) = \frac{1}{2} + o(s^{-n}) \quad (9.18)$$

for any  $s > 1$ . The second term is  $-f_2(y)(1-y)^{-1/2}$ , where  $f_2(y) = \sqrt{1-p^2(\lambda_{\text{mc}}^-)^2} y/(2y(1+p\lambda_{\text{mc}}^-))$  and we have used (9.17). From (A.7), (A.3), and (A.4), then,

$$\begin{aligned} \rho_n^{(2)} &= -f_2(1) \left(1 - \frac{1}{8n}\right) \frac{1}{\sqrt{\pi n}} - f_2'(1) \frac{1}{2n\sqrt{\pi n}} + O(n^{-5/2}) \\ &= -\frac{\sqrt{1-q}}{2} \frac{1}{\sqrt{\pi n}} + \frac{p^2 + 6q^2(1-q)}{32q^2\sqrt{1-q}} \frac{1}{n\sqrt{\pi n}} + O(n^{-5/2}) \end{aligned} \quad (9.19)$$

The third term is  $f_3(y)(1-y)^{1/2}$ , where

$$f_3(y) = \frac{\beta(p-\beta)\sqrt{1-p^2(\lambda_{\text{mc}}^-)^2} y}{2yp^2(1-\beta)(1+p\lambda_{\text{mc}}^-)(\lambda_{\text{hd}} - \lambda_{\text{mc}}^- y)} \quad (9.20)$$

As above

$$\begin{aligned} \rho_n^{(3)} &= -f_3(1) \frac{1}{2n\sqrt{\pi n}} + O(n^{-5/2}) \\ &= \frac{\sqrt{1-q}\beta(p-\beta)}{4(\beta-q)^2} \frac{1}{n\sqrt{\pi n}} + O(n^{-5/2}) \end{aligned} \quad (9.21)$$

Adding (9.18), (9.19), and (9.21) gives the density  $\rho_n$  to order  $n^{-3/2}$ :

$$\begin{aligned} \rho_n &= \frac{1}{2} + \frac{\sqrt{1-q}}{2} \frac{1}{\sqrt{\pi n}} \\ &\quad + \left( \frac{p^2 + 6q^2(1-q)}{32q^2\sqrt{1-q}} + \frac{\beta\sqrt{1-q}(p-\beta)}{4(\beta-q)^2} \right) \frac{1}{n\sqrt{\pi n}} + O(n^{-5/2}) \end{aligned} \quad (9.22)$$

In the  $p \searrow 0$  limit, with the scaling  $\alpha = p\bar{\alpha}$ ,  $\beta = p\bar{\beta}$ , this result corresponds to that determined in ref. 13. The last coefficient (9.22) is singular on the  $\beta = q$  boundary of the maximum current region and in particular at  $p = 1$ ; see the comment following (8.20).

(ii) **The low density region:**  $\alpha < \beta$  and  $\alpha < q$  (region A in Fig. 2). Here  $\lambda_0 = \lambda_{\text{id}}$ . Since  $y_{\pm} = \lambda_{\text{mc}}^{\pm}/\lambda_0 > 1$ , the square root singularities lie strictly to

the right of the pole at  $y = 1$ . Let us again write  $\rho_n = \rho_n^{(1)} + \rho_n^{(2)} + \rho_n^{(3)} + \rho_n^{(4)}$ , with  $\rho_n^{(i)}$  the contribution from the  $i$ th term in (9.13). Each of the first two terms in (9.13) has a simple pole at  $y = 1$ , so that from (A.7),

$$\begin{aligned} \rho_n^{(1)} + \rho_n^{(2)} &= \left[ \frac{2y - 1 + \lambda_{\text{ld}} p y - \sqrt{A(y)}}{2y(1 + p\lambda_{\text{ld}})} \right]_{y=1} + o(y_-^{-n}) \\ &= \frac{\alpha(1 - \alpha)}{p - \alpha^2} + o(y_-^{-n}) \end{aligned} \quad (9.23)$$

where we have used  $\sqrt{A(1)} = (p - 2\alpha + \alpha^2)/(p(1 - \alpha))$  (note that this is the positive square root). To go further in the asymptotics we must consider separately two subregions of the low density region, and their common boundary (see Fig. 2).

**Subregion AI:**  $\beta < q$ . In this subregion  $\lambda_{\text{ld}} < \lambda_{\text{hd}} < \lambda_{\text{mc}}^-$ . Since  $\beta < q$ , the pole at  $y_1 = \lambda_{\text{hd}}/\lambda_{\text{ld}} > 1$  lies on the first sheet and satisfies  $1 < y_1 < y_-$ , and thus makes the next contribution to the asymptotics beyond (9.23). Thus

$$\begin{aligned} \rho_n^{(3)} + \rho_n^{(4)} &= \left[ \frac{\beta((p - \beta)\sqrt{A(y)} + (p - \beta) - \lambda_{\text{ld}} p y(2 - \beta - p))}{2y p^2(1 + p\lambda_0)(1 - \beta)\lambda_{\text{hd}}} \right]_{y=y_1} \\ &\quad \times \left( \frac{\lambda_{\text{hd}}}{\lambda_{\text{ld}}} \right)^{-n} + o(s^{-n}) \\ &= \frac{(1 - \alpha)(p - 2\beta + \beta^2)}{(p - \alpha^2)(1 - \beta)} \left( \frac{\alpha(p - \alpha)(1 - \beta)}{\beta(p - \beta)(1 - \alpha)} \right)^{n+1} + o(s^{-n}) \end{aligned} \quad (9.24)$$

for some  $s > \lambda_{\text{hd}}/\lambda_{\text{ld}}$ . The asymptotics to order  $o(s^{-n})$  are obtained by adding (9.23) and (9.24):

$$\rho_n = \frac{\alpha(1 - \alpha)}{p - \alpha^2} + \frac{(1 - \alpha)(p - 2\beta + \beta^2)}{(p - \alpha^2)(1 - \beta)} \left( \frac{\alpha(p - \alpha)(1 - \beta)}{\beta(p - \beta)(1 - \alpha)} \right)^{n+1} + o(y_-^{-n}) \quad (9.25)$$

Further corrections, which arise from the singularity at  $y_-$ , could be calculated; the leading order is  $O(y_-^{-n}/n^{3/2})$ .

**Subregion AII:**  $\beta > q$ . Now  $\lambda_{\text{ld}} < \lambda_{\text{mc}}^-$  and  $1 < y_-$ ; the next contribution to the asymptotics beyond (9.23) comes from the square root singularity at  $y_-$ , present in the second and third terms of (9.13). Now

$$A(y) = (1 - y/y_-)(1 - y p^2 \lambda_{\text{mc}}^- \lambda_{\text{ld}}) \quad (9.26)$$

so that, writing  $\rho_n^{(2)*}$  for the contribution to  $\rho_n^{(2)}$  from this singularity, we have

$$\begin{aligned}
& \rho_n^{(2)*} + \rho_n^{(3)} \\
&= \frac{1}{1 + p\lambda_{\text{ld}}} \left[ -\frac{\sqrt{1 - yp^2\lambda_{\text{mc}}^-\lambda_{\text{ld}}}}{2y(1-y)} + \frac{\beta(p-\beta)\sqrt{1 - yp^2\lambda_{\text{mc}}^-\lambda_{\text{ld}}}}{2yp^2(1-\beta)(\lambda_{\text{hd}} - \lambda_{\text{ld}}y)} \right]_{y=y_-} \\
&\quad \times \left( -\frac{y_-^{-n}}{2n\sqrt{\pi n}} \right) + O(y_-^{-n}/n^{5/2}) \\
&= -\frac{\alpha(p-\alpha)(1-p)^{1/4}\sqrt{\lambda_{\text{mc}}^-(\lambda_{\text{ld}} - \lambda_{\text{hd}})}}{2(p-\alpha^2)(\lambda_{\text{mc}}^- - \lambda_{\text{ld}})(\lambda_{\text{mc}}^- - \lambda_{\text{hd}})} \frac{1}{n\sqrt{\pi n}} \left( \frac{\alpha(p-\alpha)}{(1-\alpha)(q+1-p)} \right)^n \\
&\quad + O\left( \frac{1}{n^2\sqrt{n}} \left( \frac{\alpha(p-\alpha)}{(1-\alpha)(q+1-p)} \right)^n \right) \tag{9.27}
\end{aligned}$$

From (9.23) and (9.27),

$$\begin{aligned}
\rho_n &= \frac{\alpha(1-\alpha)}{p-\alpha^2} - \frac{\alpha(p-\alpha)(1-p)^{1/4}\sqrt{\lambda_{\text{mc}}^-(\lambda_{\text{ld}} - \lambda_{\text{hd}})}}{2(p-\alpha^2)(\lambda_{\text{mc}}^- - \lambda_{\text{ld}})(\lambda_{\text{mc}}^- - \lambda_{\text{hd}})} \\
&\quad \times \frac{1}{n\sqrt{\pi n}} \left( \frac{\alpha(p-\alpha)}{(1-\alpha)(q+1-p)} \right)^n \\
&\quad + O\left( \frac{1}{n^2\sqrt{n}} \left( \frac{\alpha(p-\alpha)}{(1-\alpha)(q+1-p)} \right)^n \right) \tag{9.28}
\end{aligned}$$

**The AI/AII boundary:**  $\beta = q$ . Here  $y_1 = y_-$  and the leading correction to  $\rho_n$  beyond the constant term (9.23) is an  $O(n^{-1/2})$  contribution from the third term in (9.13). From (9.23) and (9.26),

$$\begin{aligned}
\rho_n &= \frac{\alpha(1-\alpha)}{p-\alpha^2} + \left[ \frac{\beta(p-\beta)\sqrt{1 - yp^2\lambda_{\text{mc}}^-\lambda_{\text{ld}}}}{2(1+p\lambda_{\text{ld}})yp^2(1-\beta)\lambda_{\text{hd}}} \right]_{y=y_1} \frac{y_1^{-n}}{\sqrt{\pi n}} + O\left( \frac{y_1^{-n}}{n\sqrt{\pi n}} \right) \\
&= \frac{\alpha(1-\alpha)}{p-\alpha^2} + \frac{\alpha(p-\alpha)(1-p)^{1/4}}{(p-\alpha^2)} \sqrt{\frac{1-\beta}{\beta(p-\beta)}} \\
&\quad \times \frac{1}{\sqrt{\pi n}} \left( \frac{\alpha(p-\alpha)}{\beta^2(1-\alpha)} \right)^n + O\left( \frac{y_1^{-n}}{n\sqrt{\pi n}} \right) \tag{9.29}
\end{aligned}$$

The low-density results (9.25), (9.28), and (9.29) agree with ref. 13 in the  $p \searrow 0$  limit.

(iii) **The high density region:**  $\beta < \alpha$  and  $\beta < q$  (region B in Fig. 2). In this region the generating function is obtained by the substitution  $\lambda_0 = \lambda_{\text{hd}}$  in (9.13). After some tedious algebra, this leads to

$$\Phi(y) = \frac{p - \beta}{p - \beta^2} \frac{1}{1 - y} \quad (9.30)$$

so that the density is constant:

$$\rho_n = \frac{p - \beta}{p - \beta^2} \quad (9.31)$$

## 10. EXACT EXPRESSIONS FOR FINITE SYSTEMS

In this section we obtain exact and explicit expressions for the current and density profile for finite systems and all values of  $\alpha$ ,  $\beta$  and  $p$ . We shall do this by using the algebraic rules (4.3) and (4.4). This provides a complementary approach to that of Sections 8 and 9 where large  $N$  properties are calculated directly.

Our first task is to evaluate  $z_n = \langle W_1 | K^n | V_1 \rangle$ . We proceed by writing  $K^n$  as a sum of irreducible (with respect to rule (4.3)) strings in the following manner

$$K^n = \sum_{r=0}^n a_{n,r} \sum_{q=0}^r E_1^{r-q} D_1^q \quad (10.1)$$

It turns out that  $a_{n,r}$  is given by the expression

$$a_{n,r} = \sum_{t=0}^{n-r} \left[ \binom{n}{r+t} \binom{n-r-1}{t} - \binom{n+1}{r+t+1} \binom{n-r-2}{t-1} \right] (1-p)^t \quad (10.2)$$

with the conventions  $\binom{X}{0} = 1$  and  $\binom{X}{-1} = 0$  for any integer  $X$ . The proof of (10.2) is left to Appendix B. Here we check a few simple cases. From (10.2) and our conventions for the binomial coefficients we find

$$\begin{aligned} a_{n,n} &= 1 \\ a_{n,n-1} &= n - (1-p) \\ a_{n,n-2} &= \binom{n}{2} - (1-p) \\ a_{n,n-3} &= \binom{n}{3} + \left[ 2 \binom{n}{2} - \binom{n+1}{2} \right] (1-p) - (1-p)^2 \end{aligned} \quad (10.3)$$

which yield using (10.1)

$$\begin{aligned}
 K &= p + (D_1 + E_1) \\
 K^2 &= p + (1 + p)(D_1 + E_1) + (D_1^2 + E_1 D_1 + E_1^2) \\
 K^3 &= 2p - p^2 + (2 + p)(D_1 + E_1) + (2 + p)(D_1^2 + E_1 D_1 + E_1^2) \\
 &\quad + (D_1^3 + E_1 D_1^2 + E_1^2 D_1 + E_1^3)
 \end{aligned} \tag{10.4}$$

as can be verified by direct calculation. From (10.2) we determine an exact expression for  $z_N$  by using the action of  $D_1, E_1$  (4.4):

$$z_N = \sum_{r=0}^N a_{N,r} \frac{(p(1-\beta)/\beta)^{r+1} - (p(1-\alpha)/\alpha)^{r+1}}{(p(1-\beta)/\beta) - (p(1-\alpha)/\alpha)} \tag{10.5}$$

where, without loss of generality, we have taken  $\langle W_1 | V_1 \rangle = 1$ . Together with (4.29), (10.5) yields an exact expression for the current.

To check the limit  $p \rightarrow 0$  we use the identity

$$\sum_{i=-\infty}^{\infty} \binom{N}{X-i} \binom{M}{Y+i} = \binom{N+M}{X+Y} \tag{10.6}$$

in order to obtain

$$a_{n,r} \rightarrow \binom{2n-r-1}{n-r} - \binom{2n-r-1}{n-r-1} = \frac{r(2n-r-1)!}{n!(n-r)!} \tag{10.7}$$

This agrees with Eq. (39) of ref. 13.

Also consider  $p = 1$ , then (10.2) becomes  $a_{n,r} = \binom{n}{r}$  and

$$z_N = \frac{(1-\beta)\beta^{-(N+1)} - (1-\alpha)\alpha^{-(N+1)}}{(1-\beta)/\beta - (1-\alpha)/\alpha} \tag{10.8}$$

One can check that this recovers the results (7.6) and (7.17) of Section 7.

In order to write down the density profile we use an expression derived in Appendix B:

$$D_1 K^n = (1-p) \sum_{r=0}^{n-1} A(r) K^{n-r} + \sum_{r=0}^n a_{n,r} D_1^{r+1} \tag{10.9}$$

where  $a_{n,r}$  is given by (10.2) and

$$A(m) = \sum_{t=0}^{m-1} \frac{1}{m} \binom{m}{t} \binom{m}{t+1} (1-p)^t \quad (10.10)$$

with the convention  $A(0) = 1$ . It can be checked that

$$pA(n-1) = a_{n,0} \quad \text{for } n > 0 \quad (10.11)$$

Inserting (10.9) into (4.31) yields

$$\begin{aligned} \langle \tau_i \rangle_N = Z_N^{-1} \times & \left[ pz_{N-1} + (1-p) \sum_{r=0}^{N-i-1} A(r) z_{N-1-r} \right. \\ & \left. + z_{i-1} \sum_{r=0}^{N-i} a_{N-i,r} (p(1-\beta)/\beta)^{r+1} \right] \end{aligned} \quad (10.12)$$

Expression (10.12) together with (10.5) gives an exact expression for the density profile of parallel updating for all system sizes. Through the mappings of Section 5 it also provides exact expressions for the density profiles of ordered and sublattice parallel updating.

The Case  $\alpha = \beta = p$ . More can be said in the special case of  $\alpha = \beta = p$  where many formulae simplify considerably. We take advantage of this to simplify the expression for the density profile and to write the two-point correlation functions as a sum of one point correlation functions.

First we note that in this case (10.5) simplifies as follows:

$$\begin{aligned} z_N &= \sum_{r=0}^n \sum_{t=r}^n \left[ \binom{n}{t} \binom{n-r-1}{t-r} - \binom{n+1}{t+1} \binom{n-r-2}{t-r-1} \right] (r+1)(1-p)^t \\ &= \sum_{t=0}^n \left[ \binom{n}{t} \binom{n+1}{t} - \binom{n+1}{t+1} \binom{n}{t-1} \right] (1-p)^t \\ &= \sum_{t=0}^n \frac{1}{n+1} \binom{n+1}{t} \binom{n+1}{t+1} (1-p)^t = A(n+1) \end{aligned} \quad (10.13)$$

where  $A(m)$  is given by (10.10) and we have used

$$\sum_{r=0}^t \binom{M-r}{t-r} (r+1) = \binom{M+2}{t} \quad (10.14)$$

From our mapping of the model onto the ASEP with a second class particle, described in Section 5, one can check that (10.13) is precisely formula (4.10) in ref. 25.



We also find using the representation (6.6)–(6.8) that

$$\begin{aligned} (D_1 + p)K - K(D_1 + p) &= K(E_1 + p) - (E_1 + p)K = D_1E_1 - E_1D_1 \\ &= (1-p) |V_1\rangle\langle W_1| \end{aligned} \quad (10.15)$$

Thus the density profile is given by

$$\begin{aligned} \langle \tau_i \rangle_N &= \langle \tau_{i+1} \rangle_N + \frac{(1-p)}{Z_N} A(i) A(N-i-1) \\ &= \frac{(1-p) \sum_{n=0}^{N-i} A(N-n) A(n) + pA(N)}{A(N+1) + pA(N)} \end{aligned} \quad (10.16)$$

One can also use (10.15) to relate the two-point correlation function (4.32) to one-point correlations:

$$\begin{aligned} \langle \tau_i(1-\tau_j) \rangle_N &= \langle \tau_i(1-\tau_{j-1}) \rangle_N + \frac{(-p)^{j-i-1}}{Z_N} A(N-j+i+1) \\ &\quad + (1-p) \frac{A(N-j+1)}{Z_N} \sum_{n=0}^{j-i-2} (-p)^n \langle \tau_i \rangle_{j-2-n} Z_{j-2-n} \end{aligned} \quad (10.17)$$

$$\begin{aligned} &= \frac{1}{Z_N} \sum_{l=i+1}^j (-p)^{l-i-1} A(N-l+i+1) \\ &\quad + \frac{(1-p)}{Z_N} \sum_{l=i+2}^j A(N-l+1) \sum_{n=0}^{l-i-2} (-p)^n \langle \tau_i \rangle_{l-2-n} Z_{l-2-n} \end{aligned} \quad (10.18)$$

## 11. THE DENSITY IN THE BULK

In this section we combine the information derived from generating functions in Sections 8 and 9 with the exact calculations of the preceding section to obtain expressions for the bulk density of the system, that is, for the large- $N$  limit of  $\langle \tau_i \rangle_N$  at constant  $\theta = i/N$ . As we will see, this bulk density is constant except on the boundary of the low and high density regions, and its value is may be guessed by taking the  $n \rightarrow \infty$  limit in the asymptotics of Section 9, that is, in (9.22), in (9.25) and (9.28), and in (9.31) (where no limit is needed). However, it needs to be shown that this limit indeed gives the correct bulk density, and we shall do so here.

The key to the calculation is to study the difference in densities at successive sites. Writing  $z_n = z_n(\alpha, \beta, p)$ ,  $z_n^* = z_n(1, \beta, p)$ , and  $z_n^\dagger = z_n(p, p, p)$ , we have from (10.12), (10.5), and (10.13),

$$\langle \tau_i \rangle_N = \frac{pz_{N-1}(\alpha, \beta, p) + (1-p) \sum_{r=0}^{N-i-1} z_{r-1}^\dagger z_{N-1-r} + [p(1-\beta)/\beta] z_{i-1} z_{N-i}^*}{z_N + pz_{N-1}} \quad (11.1)$$

and hence the density difference  $\Delta\rho_i = \langle \tau_i \rangle_N - \langle \tau_{i-1} \rangle_N$  is given by

$$\Delta\rho_i = \frac{[p(1-\beta)/\beta](z_{i-1} z_{N-i}^* - z_{i-2} z_{N-i+1}^*) - (1-p) z_{N-i-1}^\dagger z_{i-1}}{z_N + pz_{N-1}} \quad (11.2)$$

We analyze the asymptotic ( $N \rightarrow \infty$ ) behavior of (11.2) in various parts of the phase plane; properties of the bulk are obtained by the scaling  $i = \theta N$ ,  $0 < \theta < 1$ , and we will also be interested in the transition regions  $i \ll N$ ,  $N-i \ll N$ . The boundary regions in which  $i$  or  $N-i$  remains finite were investigated in Section 9.

The point  $\alpha = \beta = p$  always lies in the maximum current region, so that by (8.20),  $z_n^\dagger \sim Cn^{-3/2}(\lambda_{\text{mc}}^-)^{-n}$  for large  $n$ , where here and below  $C$  designates some unspecified constant. When  $(\alpha, \beta, p)$  lies in the maximum current region, so does  $(1, \beta, p)$ , so that  $z_n$  and  $z_n^*$  have this same asymptotic form and thus for  $N$ ,  $i$ , and  $N-i$  large,  $\Delta\rho_i \sim CN^{3/2}i^{-3/2}(N-i)^{-3/2}$ . Thus in the bulk  $\Delta\rho_i = O(N^{-3/2})$  and hence the density in the bulk is constant. Moreover,

$$\langle \tau_{N-i} \rangle_N - \rho_{\text{bulk}} = \sum_{j=\theta N}^{N-i} \Delta\rho_j \quad (11.3)$$

vanishes as  $i, N \rightarrow \infty$ , so that (9.22) implies that this bulk density has value  $1/2$ .

When  $(\alpha, \beta, \gamma)$  lies in the low density region and  $\beta < q$  (i.e., in subregion I),  $(1, \beta, p)$  lies in the high density region, and thus from (8.22) and reflection symmetry,  $z_n \sim C\lambda_{\text{ld}}^{-n}$  and  $z_n^* \sim C\lambda_{\text{hd}}^{-n}$ ; since here  $\lambda_{\text{hd}} < \lambda_{\text{mc}}^-$ ,  $\Delta\rho_i \sim C(\lambda_{\text{ld}}/\lambda_{\text{hd}})^{(N-i)}$ . As above, this implies that the bulk density is constant and equal to its value at the left end of the system, which, from (9.31) and the reflection symmetry, is  $\alpha(1-\alpha)/(p-\alpha^2)$ . The argument in subregion II is similar, with  $z_n^* \sim Cn^{-3/2}(\lambda_{\text{mc}}^-)^{-n}$  and  $\Delta\rho_i \sim C(N-i)^{-3/2}(\lambda_{\text{ld}}/\lambda_{\text{mc}}^-)^{(N-i)}$ ; the bulk density is the same. By reflection symmetry the bulk density in the high density phase is constant and equal to  $(p-\beta)/(p-\beta^2)$ .

The case  $\alpha = \beta < q$  requires special attention. Here a slight extension of the arguments of Section 8 shows that  $z_n \sim Cn\lambda_{\text{ld}}^{-n}$ . As in the subregion I case above,  $z_n^* \sim C\lambda_{\text{hd}}^{-n}$  and the  $z_n^\dagger$  term in (11.2) can be neglected; since

$\lambda_{\text{hd}} = \lambda_{\text{ld}}$ ,  $\Delta\rho_i \sim CN^{-1}$ , so that the density profile is linear. The values of  $\langle \tau_i \rangle_N$  at the left and right ends of the system are, from (9.31) and the symmetry,  $\alpha(1-\alpha)/(p-\alpha^2)$  and  $(p-\beta)/(p-\beta^2)$ , respectively. This linear profile may, as usual, be interpreted as a superposition of shocks.

## 12. SUMMARY

In this paper, we have presented an exact solution for the steady state of a simple cellular automaton describing traffic flow: the ASEP with parallel (synchronous) updating and open boundary conditions. The solution is based on recursion relations in the system size for the steady-state weights of the configurations, or, equivalently, on formulae for these weights as matrix elements of operators satisfying a quartic algebra. By writing these operators as rank four tensors, we were also able to express the relevant physical quantities in terms of a simpler matrix algebra in which the operators satisfy quadratic relations.

We used several different methods to extract explicit expressions for observables from the matrix algebra. The first applied when  $p = 1$ , in which case a two dimensional representation of the quadratic algebra cubists; this made it possible to obtain analytic expressions, in both the finite and the infinite system, for the current and for one and two point correlation functions. The results confirmed conjectures in refs. 11 and 21. The two point function is particularly interesting, because its oscillating behavior directly reflects the particle-hole attraction caused by the parallel updating. For  $\alpha = \beta$  (still with  $p = 1$ ) we obtained also closed formulae for the fluctuations in the number of particles (cars) in the system. Second, for general  $p$  we derived exact formulae, in finite systems, for the current and the one point function, and for the two point function in the case  $\alpha = \beta = p$ ; the method was essentially an inductive use of the relations of the matrix algebra. The resulting formulae involve rather complicated combinatorial expressions in which it is difficult to take the limit of infinite system size. Third, again for general  $p$ , we used the analytic properties of generating functions to compute asymptotic expressions for the current (and therefore the phase diagram) and for density profiles near the boundaries of the system. Finally, we combined the results of the last two methods to determine the density in the bulk.

Our results confirm the phase diagram conjectured in ref. 11. It is similar to that of random sequential updating:<sup>(13, 14)</sup> there are three phases and, for example, near the right boundary we found exponential decay to the bulk density in the low density phase, algebraic decay to the bulk density in the maximum current phase, and a constant density profile in the high density phase. As  $p$  increases, the portion of the phase plane corresponding to the maximum current phase shrinks until at  $p = 1$  only the

high and low density phases are present. It would be of interest to see if the phase diagram could be predicted by simple physical considerations such as those of ref. 42.

By considering mappings of the matrix algebra used here to those applicable to other updating schemes, we can directly translate all our results (finite size and asymptotics) for the current and the profiles to the case where the update of the ASEP is done in discrete time but not simultaneously (specifically, with the ordered sequential update and sublattice parallel updates). As similar mapping of algebras shows that our results apply to a system of particles on a ring, with one second class particle, in the grand canonical ensemble. Since relatively few exact properties of the discrete time updating schemes were previously known—essentially only the asymptotic current<sup>(16)</sup>—we obtain new results for these models. For example, we verify all conjectured results in Table I of ref. 11 (these describe bulk properties) and derive new formulae for density profiles both for finite systems and asymptotically. The simple translation rules for the current and one-point functions (independent of the system size or any other parameters) are surprising, since the two point function of the ASEP with parallel updating is very different from that for the other updating schemes. It would be interesting to investigate if similar relations are true not only for the ASEP but for other models.

## APPENDIX A. ASYMPTOTICS FROM GENERATING FUNCTIONS

The asymptotic behavior of the coefficients  $h_n$  of a generating function  $h(t) = \sum_{n=0}^{\infty} h_n t^n$  can frequently be determined, up to order  $o(s^{-n})$ , from knowledge of the singularities of  $h$  in a disk  $|t| < r$  with  $r > s$ . We analyze below the two cases of this sort. Recall that for any real  $\gamma$  and complex number  $t^*$  we have the following Taylor series,

$$(1 - t/t^*)^\gamma = \sum_{n=0}^{\infty} a_{\gamma,n} \left(\frac{t}{t^*}\right)^n, \quad a_{\gamma,n} = O(n^{-\gamma-1}) \quad (\text{A.1})$$

For application to Sections 8 and 9 we will need the special cases

$$a_{-1,n} = 1 \quad (\text{A.2})$$

$$a_{-1/2,n} = \frac{1}{\sqrt{\pi n}} - \frac{1}{8n\sqrt{\pi n}} + O\left(\frac{1}{n^2\sqrt{n}}\right) \quad (\text{A.3})$$

$$a_{1/2,n} = -\frac{1}{2n\sqrt{\pi n}} + O\left(\frac{1}{n^2\sqrt{n}}\right) \quad (\text{A.4})$$

*Case 1.* If the only singularities of  $h$  in the disc  $|z| < r$  are simple poles at  $t_1, \dots, t_m$  and  $c_i = \lim_{t \rightarrow t_i} (1 - t/t_i) h(t)$ , then  $h - \sum_{i=1}^m c_i (1 - t/t_i)^{-1}$  is analytic in  $|t| < r$  and hence for any  $s < r$  we have, from (A.1) and (A.2),

$$h_n = c_1 t_1^{-n} + \dots + c_m t_m^{-n} + o(s^{-n}) \quad (\text{A.5})$$

*Case 2.* Suppose that  $h(t)$  has simple poles at  $t_i$ , with  $c_i$  defined as above, as well as a power singularity at some point  $t_0 > 0$ ; we assume that  $|t_i| < t_0$  for  $i = 1, \dots, m$  and that  $h(t)$  can be written in the form

$$h(t) = g(t)(1 - t/t_0)^\gamma \quad (\text{A.6})$$

where the only singularities of  $g$  in the disk are the poles. Then for any  $k \geq 0$ ,

$$h_n = \sum_{i=0}^m c_i t_i^{-n} + \sum_{j=0}^k b_j a_{\gamma+j, n} t_0^{-n} + O(n^{-\gamma-k-2} t_0^{-n}) \quad (\text{A.7})$$

where  $b_j = g^{(j)}(t_0)(-t_0)^j/j!$ . To verify (A.7) write  $h_n = (2\pi i)^{-1} \oint_C h(t) t^{-n-1} dt$ , where the contour  $C$  has  $m+1$  components  $C_0, C_1, \dots, C_m$ . For  $i \geq 1$ ,  $C_i$  is a small circle, traced clockwise, around the point  $t_i$ , and gives the term  $c_i t_i^{-n}$  in (A.7).  $C_0$  follows the circle  $|t| = s$  counter-clockwise from just above to just below the positive real axis, then the real axis to  $t_0 + \varepsilon$  (for  $\varepsilon$  very small), then a small circle of radius  $\varepsilon$  clockwise around  $t = t_0$ , then the real axis to  $t = s$ . In evaluating the integral on  $C_0$  we choose  $k$  so that  $k + \gamma > 0$  (proving the result for such a  $k$  proves it also for smaller  $k$ ) and write  $g(t) = \sum_{j=0}^k b_j (1 - t/t_0)^j + g_{k+1}(t)(1 - t/t_0)^{k+1}$ . The contribution of  $\oint_{C_0} b_j (1 - t/t_0)^{j+\gamma} t^{-n-1} dt$  is precisely  $b_j a_{\gamma+j, n} t_0^{-n}$ . There are two terms in the remaining contribution: the integral over the large circle is  $O(s^{-n})$ , while the integral back and forth over the real axis and around the small circle is, by our choice of  $k$ , a constant multiple of  $\oint_{t_0}^s g_{k+1}(t) \times (1 - t/t_0)^{k+1+\gamma} t^{-n-1} dt$ , which is easily estimated by the saddle point method to be  $O(n^{-\gamma-k-2} t_0^{-n})$ .

## APPENDIX B. PROOF OF FORMULAS (10.2) AND (10.9)

*Proof of (10.2).* In this appendix we shall prove

$$K^n = \sum_{r=0}^n a_{n,r} \sum_{q=0}^r E_1^{r-q} D_1^q \quad (\text{B.1})$$

where  $a_{n,r}$  is given by the following expression

$$a_{n,r} = \sum_{t=0}^{n-r} \left[ \binom{n}{n-r-t} \binom{n-r-1}{t} - \binom{n+1}{n-r-t} \binom{n-r-2}{t-1} \right] (1-p)^t \quad (\text{B.2})$$

We first require a preliminary result:

$$(D_1 + p) E_1^n = (1-p)^n (D_1 + p) + \sum_{m=0}^{n-1} (1-p)^m E_1^{n-m} \quad (\text{B.3})$$

which is easy to prove by induction using (4.3): for  $n=1$  one has  $(D_1 + p) E_1 = (1-p)(D_1 + p) + E_1$ ; then assuming (B.3) and right multiplying by  $E_1$  yields

$$(D_1 + p) E^{n+1} = (1-p)^n [(1-p)(D_1 + p) + E] + \sum_{m=2}^{n+1} (1-p)^m E_1^{n-m+1} \quad (\text{B.4})$$

$$= (1-p)^{n+1} [D_1 + p] + \sum_{m=1}^{n+1} (1-p)^m E_1^{n-m+1} \quad (\text{B.5})$$

hence (B.3) is proven by induction.

Using (B.3) eventually leads to the following recursion for  $a_{n,r}$

$$a_{n+1,r} = a_{n,r-1} + \sum_{m=0}^{n-r} a_{n,r+m} (1-p)^m \quad \text{for } 1 \leq r \leq n \quad (\text{B.6})$$

$$a_{n+1,0} = \sum_{m=0}^n a_{n,m} p (1-p)^m \quad (\text{B.7})$$

$$a_{n+1,n+1} = a_{n,n} \quad (\text{B.8})$$

with boundary condition  $a_{0,0} = 1$ . To see this left multiply (B.1) by  $K$

$$K^{n+1} = (E_1 + D_1 + p) \sum_{r=0}^n a_{n,r} \sum_{q=0}^r E_1^{r-q} D_1^q \quad (\text{B.9})$$

$$\begin{aligned} &= \sum_{r=1}^{n+1} a_{n,r-1} \sum_{q=0}^{r-1} E_1^{r-q} D_1^q + \sum_{r=0}^n a_{n,r} \sum_{q=0}^r (1-p)^{r-q} [D_1 + p] D_1^q \\ &+ \sum_{r=1}^n a_{n,r} \sum_{q=0}^{r-1} \sum_{m=0}^{r-q-1} (1-p)^m E_1^{r-q-m} D_1^q \end{aligned} \quad (\text{B.10})$$

where we have relabeled the indices  $r, q$  in the first term of (B.10) and used (B.3) to generate the second two terms. Relying on not a little dexterity in relabeling and manipulating sums one can develop the second two terms of (B.10) as follows

$$\begin{aligned}
& \sum_{r=0}^n a_{n,r} \sum_{q=0}^r (1-p)^{r-q} [D_1 + p] D_1^q \\
&= \sum_{r=0}^n a_{n,r} [(1-p)^r p + D_1^{r+1}] \\
&\quad + \sum_{r=1}^n a_{n,r} \sum_{q=1}^r [(1-p)^{r+1-q} D_1^q + (1-p)^{r-q} p D_1^q] \\
&= \sum_{r=0}^n a_{n,r} [(1-p)^r p + D_1^{r+1}] + \sum_{r=1}^n a_{n,r} \sum_{q=1}^r (1-p)^{r-q} D_1^q \\
&= \sum_{r=0}^n a_{n,r} [(1-p)^r p + D_1^{r+1}] + \sum_{q=1}^n \sum_{r=0}^{n-q} a_{n,r+q} (1-p)^r D_1^q \quad (\text{B.11})
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{r=1}^n a_{n,r} \sum_{q=0}^{r-1} \sum_{m=0}^{r-q-1} (1-p)^m E_1^{r-q-m} D_1^q \\
&= \sum_{m=0}^{n-1} \sum_{r=m+1}^n \sum_{q=0}^{r-m-1} a_{n,r} (1-p)^m E_1^{r-q-m} D_1^q \\
&= \sum_{m=0}^{n-1} \sum_{r=1}^{n-m} \sum_{q=0}^{r-1} a_{n,r+m} (1-p)^m E_1^{r-q} D_1^q \\
&= \sum_{r=1}^n \sum_{q=0}^{r-1} \sum_{m=0}^{n-r} a_{n,r+m} (1-p)^m E_1^{r-q} D_1^q \quad (\text{B.12})
\end{aligned}$$

When the expressions (B.12), (B.11) are inserted back into (B.10), the second term in the square brackets of (B.11) becomes the  $q=r$  component of the first term of (B.10), and after relabeling indices the third term of (B.11) becomes the  $q=r$  component of (B.12), leading to

$$\begin{aligned}
K^{n+1} &= \sum_{r=1}^{n+1} \sum_{q=0}^r a_{n,r-1} E_1^{r-q} D_1^q \\
&\quad + \sum_{r=1}^n \sum_{q=0}^{r-1} \sum_{m=0}^{n-r} a_{n,r+m} (1-p)^m E_1^{r-q} D_1^q + \sum_{m=0}^n p(1-p)^m a_{n,m} \quad (\text{B.13})
\end{aligned}$$

From (B.13) one can read off (B.6)–(B.8).

Now assume that  $a_{n,r}$  can be written as

$$a_{n,r} = \sum_{t=0}^{n-r} d_{n,r,t} (1-p)^t \quad (\text{B.14})$$

Inserting (B.14) into (B.6), (B.7) and (B.8) respectively yields

$$d_{n+1,r,t} = d_{n,r-1,t} + \sum_{m=0}^t d_{n,r+m,t-m} \quad \text{for } 1 \leq r \leq N \quad (\text{B.15})$$

$$d_{n+1,0,t} = \sum_{m=0}^t d_{n,m,t-m} - \sum_{m=0}^{t-1} d_{n,m,t-1-m} \quad (\text{B.16})$$

$$d_{n,n,t} = \delta_{t,0} \quad (\text{B.17})$$

In order to show that

$$d_{n,r,t} = \binom{n}{r+t} \binom{n-r-1}{t} - \binom{n+1}{r+t+1} \binom{n-r-2}{t-1} \quad (\text{B.18})$$

satisfies (B.15)–(B.17) we employ two well known identities

$$\sum_{i=0}^{N-M} \binom{N-i}{M-i} = \binom{N+1}{M} \quad (\text{B.19})$$

$$\binom{N}{M} = \binom{N-1}{M} + \binom{N-1}{M-1} \quad (\text{B.20})$$

Using (B.19) yields

$$\sum_{m=0}^t d_{n,r+m,t-m} = \binom{n}{r+t} \binom{n-r}{t} - \binom{n+1}{r+t+1} \binom{n-r-1}{t-1} \quad (\text{B.21})$$

Then (B.20) becomes

$$\begin{aligned} d_{n,r-1,t} + \sum_{m=0}^t d_{n,r+m,t-m} \\ = \binom{n+1}{r+t} \binom{n-r}{t} - \binom{n+2}{r+t+1} \binom{n-r-1}{t-1} \end{aligned} \quad (\text{B.22})$$



which is the expression (B.18) required to satisfy (B.15). Similarly with the aid of (B.19) the repeated use of (B.20) one finds

$$\begin{aligned} \sum_{m=0}^t d_{n,m,t-m} - \sum_{m=0}^{t-1} d_{n,m,t-1-m} \\ = \binom{n}{t} \binom{n}{t} - \binom{n+1}{t+1} \binom{n-1}{t-1} - \binom{n}{t-1} \binom{n}{t-1} + \binom{n+1}{t} \binom{n-1}{t-2} \\ = \binom{n+1}{t} \binom{n}{t} - \binom{n+2}{t+1} \binom{n-1}{t-1} \end{aligned} \tag{B.23}$$

thus satisfying (B.16) when  $d_{n,r,t}$  is given by (B.18). Finally when the conventions  $\binom{X}{0} = 1$  and  $\binom{X}{-1} = 0$  for all integers  $X$  are imposed, (B.18) implies

$$d_{n,n,t} = \binom{n+1}{n+1+t} \binom{-1}{t} - \binom{n+1}{-t-1} \binom{-2}{t-1} = \delta_{t,0} \tag{B.24}$$

thus satisfying (B.17).

*Proof of (10.9).* Here we prove

$$D_1 K^n = (1-p) \sum_{r=0}^{n-1} A(r) K^{n-r} + \sum_{r=0}^n a_{n,r} D_1^{r+1} \tag{B.25}$$

First we note

$$D_1^n [E_1 + p] = (1-p)^n [E_1 + p] + \sum_{m=0}^{n-1} (1-p)^m D_1^{n-m} \tag{B.26}$$

which is proven in a similar fashion to (B.3).

To prove (B.25) by induction, one can check the case  $n=0$  or  $n=1$ , then right multiply the rhs of (B.25) by  $K$ , using (B.26) to obtain

$$\begin{aligned} D_1 K^{n+1} &= (1-p) \sum_{r=0}^{n-1} A(r) K^{n+1-r} + \sum_{r=0}^n a_{n,r} D_1^{r+1} [D_1 + E_1 + p] \\ &= (1-p) \sum_{r=0}^{n-1} A(r) K^{n+1-r} + \sum_{r=0}^n a_{n,r} D_1^{r+2} \\ &\quad + \sum_{r=0}^n a_{n,r} (1-p)^{r+1} (E_1 + p) \\ &\quad + \sum_{r=0}^n \sum_{m=0}^r a_{n,r} (1-p)^m D_1^{r+1-m} \end{aligned} \tag{B.27}$$

The third term of (B.27) becomes using (B.7)

$$\sum_{r=0}^n a_{n,r}(1-p)^{r+1}(E_1+p) = \frac{1-p}{p} a_{n+1,0}(E_1+p) \quad (\text{B.28})$$

The fourth term of (B.27) may be developed as follows

$$\begin{aligned} & \sum_{r=0}^n \sum_{m=0}^r a_{n,r}(1-p)^m D_1^{r+1-m} \\ &= \sum_{m=0}^n \sum_{r=0}^{n-m} a_{n,r+m}(1-p)^m D_1^{r+1} \\ &= \sum_{r=0}^n \sum_{m=0}^{n-r} a_{n,r+m}(1-p)^m D_1^{r+1} \\ &= \frac{a_{n+1,0}}{p} D_1 + \sum_{r=1}^n [a_{n+1,r} - a_{n,r-1}] D_1^{r+1} \end{aligned} \quad (\text{B.29})$$

where (B.6), (B.7) have been used to obtain the final equality. Putting (B.27), (B.28) and (B.29) together yields

$$\begin{aligned} D_1 K^{n+1} &= (1-p) \sum_{r=0}^{n-1} A(r) K^{n+1-r} + \frac{1-p}{p} a_{n+1,0} K + a_{n+1,0} D_1 \\ &\quad + \sum_{r=1}^n a_{n+1,r} D_1^{r+1} + D_1^{n+2} \\ &= (1-p) \sum_{r=0}^n A(r) K^{n+1-r} + \sum_{r=0}^{n+1} a_{n+1,r} D_1^{r+1} \end{aligned} \quad (\text{B.30})$$

which agrees with (B.25), thereby proving (10.9) by induction.

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